Tracking how dependently-typed functions use their arguments

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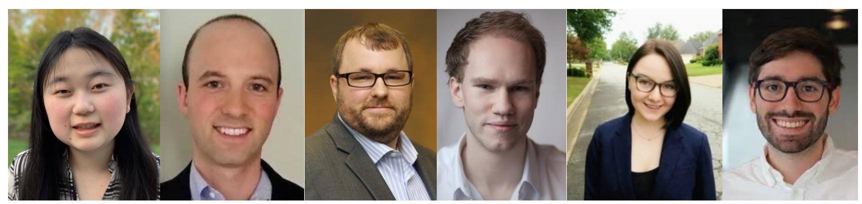

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Let's talk about constant functions

```
id : ∀ (A : Type) → A → A
id = λ A x. x
```

```
id = λ _ x. x
```

Erasure semantics for type polymorphism

Erasure semantics for type polymorphism

```
data List (A : Type) : Type where
  Nil  : List A
  Cons : A → List A → List A

map : ∀ (A B : Type) → (A → B) → List A → List B
map = λ A B f xs.
        case xs of
          Nil ⇒ Nil
          Cons y ys ⇒ Cons (f y) (map A B f ys)
```

Erasure semantics for type polymorphism

```
data
  Nil
  Cons


map = λ _ _ f xs.
        case xs of
          Nil ⇒ Nil
          Cons y ys ⇒ Cons (f y) (map _ _ f ys)
```

Erasure in **dependently-typed** languages

```
data Vec (n:Nat) (A:Type) : Type where
  Nil  : Vec Zero A
  Cons : Π(m:Nat) → A → (Vec m A) → Vec (Succ m) A

map : ∀(A B : Type) → Π(n : Nat) → (A → B)
    → Vec n A → Vec n B
map = λ A B n f v.
        case v of
          Nil ⇒ Nil
          Cons m x xs ⇒
              Cons m (f x) (map A B m f xs)
```

Erasure in **dependently-typed** languages

```
data Vec (n:Nat) (A:Type) : Type where
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map : ∀(A B : Type) → ∀(n : Nat) → (A → B)
    → Vec n A → Vec n B
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```

Erasure in **dependently-typed** languages

```
data
  Nil
  Cons



map = λ _ _ n f v.
        case v of
          Nil ⇒ Nil
          Cons _ x xs ⇒
              Cons _ (f x) (map _ _ _ f xs)
```

Refinement/Subset types

```
type EvenNat = { n : Nat | isEven n }
```

Erasable proof

```
plusIsEven : Π(m n : Nat) → isEven m → isEven n
           → isEven (m + n)
plusIsEven = λ m n p1 p2. …

plus : EvenNat → EvenNat → EvenNat
plus = λ en em. case en, em of
          (n, np), (m, mp) ⇒ (n + m, plusIsEven n m np mp)
```

Refinement/Subset types

```
plus = λ en em. case en, em of
          (n, _ ), (m, _ ) ⇒ (n + m, _)
```

Erasable code is irrelevant

- Not all terms are needed for computation: some function arguments and data structure components are there only for type checking
- Especially common in dependently-typed programming and proving
- Can call such code *irrelevant*

Why care about irrelevance?

1. The compiler can produce faster code
   – Erase arguments and their computation

Run-time irrelevance

2. The type checker can run more quickly
   – Comparing types for equality requires reduction, which can be sped up by erasure
3. Verification is less work for programmers
   – Proving that terms are equal is easier when you can ignore the irrelevant parts
4. More programs type check
   – May not be able to prove the irrelevant parts equal

Compile-time irrelevance

Less work for verification: proof irrelevance

```
type EvenNat = { n : Nat | isEven n }

-- prove equality of two EvenNats
congEvenNat : (n m : Nat)
            → (np : isEven n)
            → (mp : isEven m)
            → (n = m)
            -- no need for proof of np = mp
            → ((n, np) = (m, mp) : EvenNat)
congEvenNat = λ n m en em p. …
```

More programs type check

…when more terms are equal, *by definition*

type tells us that f is a constant function

```
example : ∀(f : ∀ (x : Bool) -> Bool)
        → (f True = f False)
example = λ f . Refl
```

Sound for the type checker to decide that these terms are equal

Proof of equality comes directly from type checker

Type checkers for dependently-typed languages should identify irrelevant code

But how?

How should type checkers for dependently-typed languages identify and take advantage of irrelevant code?

1. **Erasure**
2. **Modes**
3. **Dependency**

Core dependent type system

$$\boxed{\Gamma \vdash a : A}$$

Var

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}$$

Pi

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x : A \vdash B : \star}{\Gamma \vdash \Pi\, x{:}A.B : \star}$$

Abs

$$\frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash \Pi\, x{:}A.B : \star}{\Gamma \vdash \lambda x{:}A.\, b : \Pi\, x{:}A.B}$$

App

$$\frac{\Gamma \vdash b : \Pi\, x{:}A.B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\; a : B[a/x]}$$

Conv

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : \star \qquad \vdash A \equiv B}{\Gamma \vdash a : B}$$

Erasure

You can't use something that is not there

Miquel, TLCA 01
Barras and Bernardo, FoSSaCS 2008

Zombie/Trellys [Kimmel et al. MSFP 2012]
Dependent Haskell [Weirich et al. ICFP 2018]

ICC: Implicit Calculus of Constructions

- Extend core language with irrelevant (implicit) abstractions

$$\textsc{E-Abs}\quad \frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash \forall x : A.\, B : \star \qquad x \notin \mathsf{fv}|b|}{\Gamma \vdash \lambda x :^{\mathbf{I}} A.\, b : \forall x : A.\, B}$$

$$\textsc{E-App}\quad \frac{\Gamma \vdash b : \forall x : A.\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a^{\mathbf{I}} : B[a/x]}$$

ICC: Implicit Calculus of Constructions

- Extend core language with irrelevant (implicit) abstractions
- Annotations enable decidable type checking

E-Abs

$$\frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash \forall x : A.\, B : \star \qquad x \notin \mathsf{fv}|b|}{\Gamma \vdash \lambda x :^{\mathbf{I}} A.\, b : \forall x : A.\, B}$$

E-App

$$\frac{\Gamma \vdash b : \forall x : A.\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a^{\mathbf{I}} : B[a/x]}$$

ICC: Implicit Calculus of Constructions

- Extend core language with irrelevant (implicit) abstractions
- Annotations enable decidable type checking
- Irrelevant parameters must not appear *relevantly*
- Erasure operation $|a|$ removes irrelevant terms

$$\textsc{E-Abs}\quad \frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash \forall x{:}A.\,B : \star \qquad x \notin \mathsf{fv}|b|}{\Gamma \vdash \lambda x :^{\mathbf{I}} A.\, b : \forall x{:}A.\,B}$$

$$\textsc{E-App}\quad \frac{\Gamma \vdash b : \forall x{:}A.\,B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a^{\mathbf{I}} : B[a/x]}$$

Erasure during conversion

- Conversion between *erased* types
- Compile-time irrelevance: erased parts ignored when comparing types for equality

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : \star \qquad \vdash |A| \equiv |B|}{\Gamma \vdash a : B}\ \textsc{E-Conv-Ann}$$

Erasure: Implicit Calculus of Constructions

- Benefits
  - Simple!
  - Directly connects to erasure in compilation
  - Orthogonal: features independent from the rest of the system

- Drawbacks
  - No irrelevant projections

Irrelevant projections

Length is not statically known
and irrelevant

```
filter : ∀(A:Type) → ∀(n:Nat)
       → (A → Bool) → (Vec n A) → ∃(m:Nat) × (Vec m A)
filter = λ A n f v.
    case v of
      Nil ⇒
         (0, Nil)
      Cons m x xs ⇒
         let p = filter A m f xs in
         if f x
            then (Succ p.1, Cons p.1 x p.2)
            else p
```

UNSOUND addition to
compile-time erasure
| Vec p.1 A | = ????

Modes

Distinguish relevant and irrelevant abstractions through *modes*

Pfenning, LICS 01
Mishra-Linger and Sheard, FoSSaCS 08
Abel and Scherer, LMCS 12

DDC, Choudhury and Weirich, ESOP 22
DE, Liu and Weirich, ICFP 23

Modal types and modes

- Modal type marks irrelevant code: $\Box A$
- Type system controlled by modes: $m ::= \mathbf{R} \mid \mathbf{I}$
  - Variables have modes, must be $\mathbf{R}$ when used

    $$\Gamma ::= \varepsilon \mid \Gamma,\ x :^{m} A$$

  - Resurrection (Γ^{m}): replaces all m tags with R
  - Mode-annotated quantification:
    $\Pi x{:}^{m} A.B$ unifies $\Pi x{:}A.B$ and $\forall x{:}A.B$

Modal types for irrelevance

Only relevant variables can be used

M-Var
$$\frac{x :^{\mathbf{R}} A \in \Gamma}{\Gamma \vdash x : A}$$

M-Box
$$\frac{\Gamma^{\mathbf{I}} \vdash a : A}{\Gamma \vdash \mathbf{box}\, a : \Box A}$$

M-LetBox
$$\frac{\Gamma \vdash a : \Box A \qquad \Gamma, x :^{\mathbf{I}} A \vdash b : B \qquad \Gamma \vdash B : \star}{\Gamma \vdash \mathbf{unbox}\, x = a \,\mathbf{in}\, b : B}$$

Modal types mark irrelevant subterms.
Resurrection means that any variable can be used inside a box.

The contents of the box are accessible only in other boxes.

Mode-annotated functions

Only relevant variables can be used

M-Var

$$\frac{x :^{\mathbf{R}} A \in \Gamma}{\Gamma \vdash x : A}$$

Π-bound variables always relevant in the type

M-Pi

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x :^{\mathbf{R}} A \vdash B : \star}{\Gamma \vdash \Pi\, x :^{m} A.B : \star}$$

Types checked with "resurrected" context

M-Abs

$$\frac{\Gamma^{\mathbf{I}} \vdash \Pi\, x :^{m} A.B : \star \qquad \Gamma, x :^{m} A \vdash b : B}{\Gamma \vdash \lambda x :^{m} A.\, a : \Pi\, x :^{m} A.B}$$

Mode on Π-type determines mode in the context

M-App

$$\frac{\Gamma \vdash b : \Pi\, x :^{m} A.B \qquad \Gamma^{m} \vdash a : A}{\Gamma \vdash b\ a^{m} : B[a/x]}$$

Irrelevant arguments checked with resurrected context

M-Conv

$$\frac{\Gamma \vdash a : A \qquad \vdash A \equiv B}{\Gamma \vdash a : B}$$

Conversion ignores irrelevant arguments

Compile-time irrelevance

- Usual rules, plus
  - compare arguments marked $\mathbf{R}$
  - ignore arguments marked $\mathbf{I}$ or inside a box

$$\textsc{Eq-Rel}\quad \frac{\vdash b_1 \equiv b_2 \qquad \boxed{\vdash a_1 \equiv a_2}}{\vdash b_1\ {a_1}^{\mathbf{R}} \equiv b_2\ {a_2}^{\mathbf{R}}}$$

$$\textsc{Eq-Irr}\quad \frac{\vdash b_1 \equiv b_2}{\vdash b_1\ {a_1}^{\mathbf{I}} \equiv b_2\ {a_2}^{\mathbf{I}}}$$

Modes for irrelevance

- Benefits
  - Modes identify patterns in the semantics: don't need two different function types
  - Easy implementation: mark variables when introduced in the context, mark the context for resurrection
- Drawbacks
  - Still no irrelevant projections
  - Formation rule for Π-types looks a bit strange

- Conjecture: equivalent to ICC*

Alternative rule for Π-types

- From [Pfenning 99] [Abel and Scherer 2012]
- But: irrelevant arguments must be irrelevant *everywhere*, including in types. No parametric polymorphism!

M-Pi

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x :^{\mathbf{R}} A \vdash B : \star}{\Gamma \vdash \Pi\, x :^{m} A.B : \star}$$

M-Pi-Alt

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x :^{m} A \vdash B : \star}{\Gamma \vdash \Pi\, x :^{m} A.B : \star}$$

Dependency

Track when outputs depend on inputs

DCC, Abadi et al., POPL 99

DDC, Choudhury and Weirich, ESOP 22

DCOI, Liu, Chan, Shi, Weirich, POPL 24

Dependency tracking

- Typing judgment ensures that low-level outputs do not depend on high-level inputs

$$x :^{H} Bool \vdash a :^{L} Int$$

Input level
x can only be used when observer level is $\geq H$

Observer level
a can only use variables whose levels are $\leq L$

- Type system parameterized by ordered set of levels
  - Relevance (R < I)
  - *Other examples*: Security levels (Low < Med < High)
    Staged computation (0 < 1 < 2…)

Typing rules with dependency levels $\boxed{\Gamma \vdash a :^{\ell} A}$

Variable usage restricted by observer level

D-Var

$$\frac{x :^{m} A \in \Gamma \qquad m \leq \ell}{\Gamma \vdash x :^{\ell} A}$$

D-Pi

Vars have same level in terms and types

$$\frac{\Gamma \vdash A :^{\ell} \star \qquad \Gamma, x :^{m} A \vdash B :^{\ell} \star}{\Gamma \vdash \Pi\, x :^{m} A.B :^{\ell} \star}$$

D-Abs

Terms do not observe types, so level unimportant

$$\frac{\Gamma, x :^{m} A \vdash b :^{\ell} B \qquad \Gamma \vdash \Pi\, x :^{m} A.B :^{\ell_1} \star}{\Gamma \vdash \lambda x :^{m} A.\, b :^{\ell} \Pi\, x :^{m} A.B}$$

Π-types record the dependency levels of their arguments

D-App

$$\frac{\Gamma \vdash b :^{\ell} \Pi\, x :^{m} A.B \qquad \Gamma \vdash a :^{m} A}{\Gamma \vdash b\, a^{m} :^{\ell} B[a/x]}$$

Application requires compatible dependency levels

DCOI: irrelevant projections

Type is checked with I-observer

```
vfilter : (A:ᴵ Type) → (n:ᴵ Nat)
        → (A → Bool) → (Vec n A) → (m:ᴵ Nat) × (Vec m A)
vfilter = λ A n f v.
    case v of
      Nil ⇒ (0ᴵ, Nil)
      Cons mᴵ x xs ⇒
         let p = vfilter Aᴵ mᴵ f xs in
         if f x
            then ((Succ p.1)ᴵ, Cons p.1ᴵ x p.2)
            else p
```

Definition checks with R observer, but contains I-marked subterms

First projection allowed in I-marked subterms only

Indistinguishability: indexed definitional equality

$$\vdash a \equiv^{\ell} b$$

Observer at level ℓ cannot distinguish between terms

If observer has a higher level than the argument, arguments must agree

$$\textsc{Eq-D-App-Dist}\quad \frac{\vdash b_0 \equiv^{\ell} b_1 \qquad \vdash a_0 \equiv^{\ell} a_1 \qquad \ell_0 \leq \ell}{\vdash b_0\ {a_0}^{\ell_0} \equiv^{\ell} b_1\ {a_1}^{\ell_0}}$$

If observer does not have a higher level, arguments are ignored

$$\textsc{Eq-D-App-Indist}\quad \frac{\vdash b_0 \equiv^{\ell} b_1 \qquad \ell_0 \not\leq \ell}{\vdash b_0\ {a_0}^{\ell_0} \equiv^{\ell} b_1\ {a_1}^{\ell_0}}$$

Conversion can be used at **any** observer level

$$\frac{\textsc{D-Conv} \quad \Gamma \vdash a :^{\ell} A \qquad \Gamma \vdash B :^{\ell_0} \star \qquad \vdash A \equiv^{\ell_0} B}{\Gamma \vdash a :^{\ell} B}$$

Type system is **sound** because we **cannot** equate types with different head forms at *any* dependency level

DCOI: Dependent Calculus of Indistinguishability

- **Yiu, Chan, Shi and Weirich. *Internalizing Indistinguishability with Dependent Types.* POPL 2024**
  - Based on Pure Type System (PTS)
  - Key results: Syntactic type soundness, noninterference
- **Yiu, Chan and Weirich. *Consistency of a Dependent Calculus of Indistinguishability.* POPL 2025**
  - Predicative universe hierarchy
  - Observer-indexed propositional equality, J-eliminator
  - Key results: Consistency, normalization, and decidable (observer-indexed) equality
- All results mechanized using Rocq proof assistant

DCOI: Proof-specific features

False-elimination

D-Absurd

$$\frac{\Gamma \vdash a :^{m} \bot \qquad \Gamma \vdash A :^{\ell_1} \star}{\Gamma \vdash \mathbf{absurd}\ a :^{\ell} A}$$

Proof of false can be eliminated at *any* observer level.

Propositional indistinguishability

D-Eq

$$\frac{\Gamma \vdash a :^{m} A \qquad \Gamma \vdash b :^{m} A \qquad m \leq \ell}{\Gamma \vdash a =^{m} b :^{\ell} \star}$$

Equality must be observable to the type

D-Refl

$$\frac{\Gamma \vdash a :^{m} A}{\Gamma \vdash \mathbf{refl} :^{\ell} a =^{m} a}$$

Can create reflexivity proofs about any level

Level of proof must be less than current level

D-J

$$\frac{\begin{array}{c}\Gamma \vdash a :^{\ell_0} a_1 =^{m} a_2 \\ \Gamma, x :^{m} A, y :^{\ell_0} a_1 =^{m} x \vdash B :^{m} \star \\ \Gamma \vdash b :^{\ell} B[a_1/x][\mathbf{refl}/y] \qquad \ell_0 \leq \ell\end{array}}{\Gamma \vdash \mathbf{J}\ a\ b :^{\ell} B[a_2/x][a/y]}$$

Equality between terms at level m witnessed via substitution at level m

Equality between proofs at level ℓ_0 witnessed via substitution at level ℓ_0

DCOI: Dependent Calculus of Indistinguishability

- Benefits
  - Irrelevant projection is sound!
  - Irrelevant absurdity is sound!
  - Can reason about *indistinguishability* as a proposition
- Future work
  - Compatibility with type-directed equality (relational model)
  - Language ergonomics
    - Dependency level inference?
    - Dependency level quantification?
  - Applications besides irrelevance?

Related work on Irrelevance

- Erasure-based
  - Miquel 2001, Barras & Bernardo 2008
- Mode-based
  - Pfenning 2001, Mishra-Linger & Sheard 2008, Abel & Scherer 2012
- Dependency tracking
  - Type theory in color: Bernardy & Moulin 2013
  - Two level type theories: Kovács 2022, Annenkov et al. 2023
- Proof irrelevance: Prop/sProp (Rcoq), Prop (Agda)
  - Hofmann& Streicher 1988, Gilbert et al. 2019
- Quantitative Type Theory (Run-time irrelevance / erasure)
  - McBride 2016, Atkey 2018, Abel & Bernardy 2020,
    Choudhury et al. 2021, Moon et al. 2021, Abel et al. 2023

Conclusions

- In dependent type systems, identifying irrelevant computations is important for *efficiency* and *expressivity*
- Type systems can track more than "types", they can also tell us what happens during computation
- Dependency analysis is a powerful hammer in type system design