Chapter 15

Isometries, Local Isometries, Riemannian Coverings and Submersions, Killing Vector Fields

The goal of this chapter is to understand the behavior of isometries and local isometries, in particular their action on geodesics.

15.1 Isometries and Local Isometries

Recall that a *local isometry* between two Riemannian manifolds M and N is a smooth map $\varphi\colon M \to N$ so that

$$\langle (d\varphi)_p(u), (d\varphi_p)(v)\rangle_{\varphi(p)} = \langle u, v\rangle_p,$$

for all $p \in M$ and all $u, v \in T_pM$. An *isometry* is a local isometry and a diffeomorphism.

By the inverse function theorem, if $\varphi\colon M \to N$ is a local isometry, then for every $p \in M$, there is some open subset $U \subseteq M$ with $p \in U$ so that $\varphi \restriction U$ is an isometry between U and $\varphi(U)$.

Also recall that if $\varphi\colon M \to N$ is a diffeomorphism, then for any vector field X on M, the vector field $\varphi_* X$ on N (called the *push-forward* of X) is given by

$$(\varphi_* X)_q = d\varphi_{\varphi^{-1}(q)} X(\varphi^{-1}(q)), \qquad \text{for all } q \in N,$$

or equivalently, by

$$(\varphi_* X)_{\varphi(p)} = d\varphi_p X(p), \qquad \text{for all } p \in M.$$

Proposition 15.1. *For any smooth function $h\colon N \to \mathbb{R}$, for any $q \in N$, we have*

$$(\varphi_* X)(h)_q = X(h \circ \varphi)_{\varphi^{-1}(q)},$$

or equivalently

$$(\varphi_* X)(h)_{\varphi(p)} = X(h \circ \varphi)_p. \qquad (*)$$

It is natural to expect that isometries preserve all "natural" Riemannian concepts and this is indeed the case. We begin with the Levi-Civita connection.

Proposition 15.2. *If $\varphi\colon M \to N$ is an isometry, then*

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_* X}(\varphi_* Y), \qquad \text{for all } X, Y \in \mathfrak{X}(M),$$

where $\nabla_X Y$ is the Levi-Civita connection induced by the metric on M and similarly on N.

As a corollary of Proposition 15.2, the curvature induced by the connection is preserved; that is

$$\varphi_* R(X, Y)Z = R(\varphi_* X, \varphi_* Y)\varphi_* Z,$$

as well as the parallel transport, the covariant derivative of a vector field along a curve, the exponential map, sectional curvature, Ricci curvature and geodesics.

Actually, all concepts that are local in nature are preserved by local diffeomorphisms!

So, except for the Levi-Civita connection and the Riemann tensor on vectors, all the above concepts are preserved under local diffeomorphisms.

Proposition 15.3. *If $\varphi\colon M \to N$ is a local isometry, then the following concepts are preserved:*

(1) The covariant derivative of vector fields along a curve γ; that is

$$d\varphi_{\gamma(t)} \frac{DX}{dt} = \frac{D\varphi_* X}{dt},$$

for any vector field X along γ, with $(\varphi_* X)(t) = d\varphi_{\gamma(t)} Y(t)$, for all t.

(2) Parallel translation along a curve. If P_γ denotes parallel transport along the curve γ and if $P_{\varphi\circ\gamma}$ denotes parallel transport along the curve $\varphi \circ \gamma$, then

$$d\varphi_{\gamma(1)} \circ P_\gamma = P_{\varphi\circ\gamma} \circ d\varphi_{\gamma(0)}.$$

(3) *Geodesics. If* γ *is a geodesic in* M*, then* $\varphi \circ \gamma$ *is a geodesic in* N*. Thus, if* γ_v *is the unique geodesic with* $\gamma(0) = p$ *and* $\gamma_v'(0) = v$*, then*

$$\varphi \circ \gamma_v = \gamma_{d\varphi_p v},$$

wherever both sides are defined. Note that the domain of $\gamma_{d\varphi_p v}$ *may be strictly larger than the domain of* γ_v*. For example, consider the inclusion of an open disc into* $\mathbb{R}^2$*.*

(4) *Exponential maps. We have*

$$\varphi \circ \exp_p = \exp_{\varphi(p)} \circ d\varphi_p,$$

wherever both sides are defined. See Figure 15.1.

(5) *Riemannian curvature tensor. We have*

$$d\varphi_p R(x, y)z = R(d\varphi_p x, d\varphi_p y)d\varphi_p z, \quad \text{for all } x, y, z \in T_pM.$$

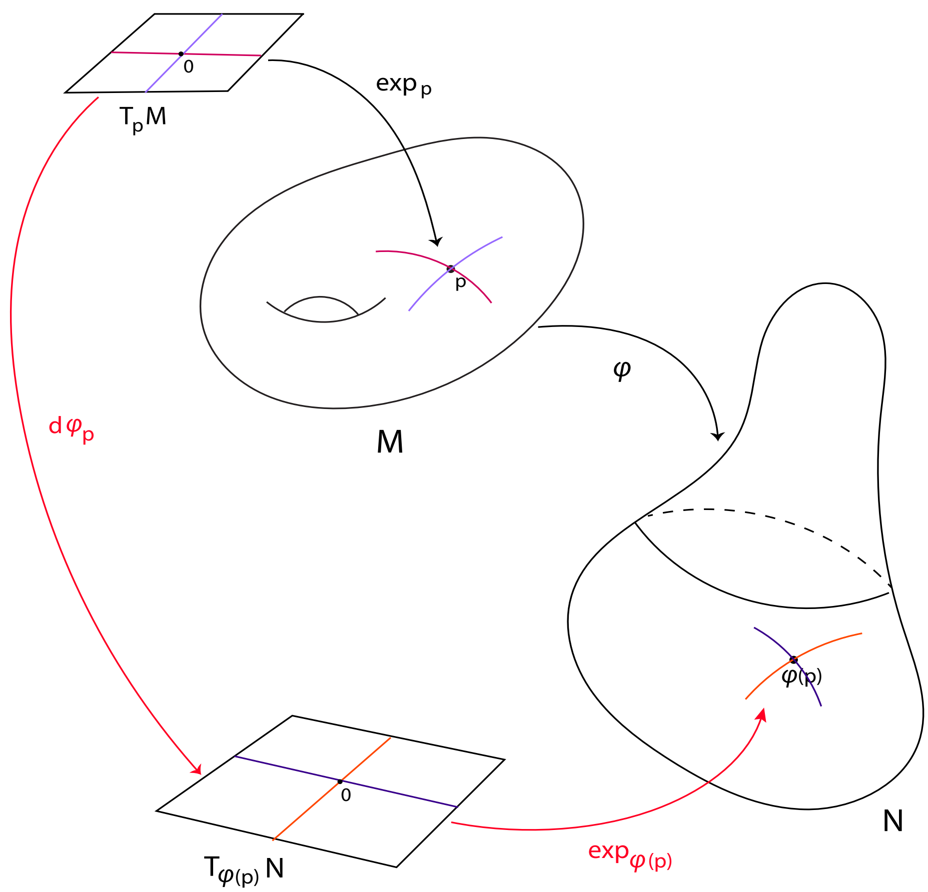

Figure 15.1: An illustration of $\varphi \circ \exp_p = \exp_{\varphi(p)} \circ d\varphi_p$. The composition of the black maps agrees with the composition of the red maps.

(6) Sectional, Ricci, and Scalar curvature. We have

$$K(d\varphi_p x, d\varphi_p y) = K(x, y)_p,$$

for all linearly independent vectors $x, y \in T_pM$*;*

$$\mathrm{Ric}(d\varphi_p x, d\varphi_p y) = \mathrm{Ric}(x, y)_p$$

for all $x, y \in T_pM$*;*

$$S_M = S_N \circ \varphi.$$

where S_M *is the scalar curvature on* M *and* S_N *is the scalar curvature on* N*.*

A useful property of local diffeomorphisms is stated below. For a proof, see O'Neill [38] (Chapter 3, Proposition 62):

Proposition 15.4. *Let* $\varphi, \psi \colon M \to N$ *be two local isometries. If* M *is connected and if* $\varphi(p) = \psi(p)$ *and* $d\varphi_p = d\psi_p$ *for some* $p \in M$*, then* $\varphi = \psi$*.*

15.2 Riemannian Covering Maps

The notion of covering map discussed in Section 9.2 (see Definition 9.2) can be extended to Riemannian manifolds.

Definition 15.1. If M and N are two Riemannian manifold, then a map $\pi\colon M \to N$ is a *Riemannian covering* iff the following conditions hold:

(1) The map π is a smooth covering map.

(2) The map π is a local isometry.

Recall from Section 9.2 that a covering map is a local diffeomorphism.

A way to obtain a metric on a manifold M is to pull-back the metric g on a manifold N along a local diffeomorphism $\varphi\colon M \to N$ (see Section 11.2).

If φ is a covering map, then it becomes a Riemannian covering map.

Proposition 15.5. *Let $\pi\colon M \to N$ be a smooth covering map. For any Riemannian metric g on N, there is a unique metric π^*g on M, so that π is a Riemannian covering.*

In general, if $\pi\colon M \to N$ is a smooth covering map, a metric on M does not induce a metric on N such that π is a Riemannian covering.

However, if N is obtained from M as a quotient by some suitable group action (by a group G) on M, then the projection $\pi\colon M \to M/G$ is a Riemannian covering.

Because a Riemannian covering map is a local isometry, we have the following useful result.

Proposition 15.6. *Let $\pi\colon M \to N$ be a Riemannian covering. Then, the geodesics of (M, g) are the projections of the geodesics of (N, h) (curves of the form $\pi \circ \gamma$, where γ is a geodesic in N), and the geodesics of (N, h) are the liftings of the geodesics of (M, h) (curves γ in N such that $\pi \circ \gamma$ is a geodesic of (M, h)).*

As a corollary of Proposition 15.5 and Theorem 9.13, every connected Riemannian manifold M has a simply connected covering map $\pi\colon \widetilde{M} \to M$, where π is a Riemannian covering.

Furthermore, if $\pi\colon M \to N$ is a Riemannian covering and $\varphi\colon P \to N$ is a local isometry, it is easy to see that its lift $\widetilde{\varphi}\colon P \to M$ is also a local isometry.

In particular, the deck-transformations of a Riemannian covering are isometries.

In general, a local isometry is not a Riemannian covering. However, this is the case when the source space is complete.

Proposition 15.7. *Let* $\pi\colon M \to N$ *be a local isometry with* N *connected. If* M *is a complete manifold, then* π *is a Riemannian covering map.*

15.3 Riemannian Submersions

Let $\pi\colon M \to B$ be a submersion between two Riemannian manifolds (M, g) and (B, h).

For every $b \in B$ in the image of π, the fibre $\pi^{-1}(b)$ is a Riemannian submanifold of M, and for every $p \in \pi^{-1}(b)$, the tangent space $T_p\pi^{-1}(b)$ to $\pi^{-1}(b)$ at p is $\mathrm{Ker}\, d\pi_p$.

The tangent space T_pM to M at p splits into the two components

$$T_pM = \mathrm{Ker}\, d\pi_p \oplus (\mathrm{Ker}\, d\pi_p)^{\perp},$$

where $\mathcal{V}_p = \mathrm{Ker}\, d\pi_p$ is the *vertical subspace* of T_pM and $\mathcal{H}_p = (\mathrm{Ker}\, d\pi_p)^{\perp}$ (the orthogonal complement of $\mathcal{V}_p$ with respect to the metric g_p on T_pM) is the *horizontal subspace* of T_pM.

Any tangent vector $u \in T_pM$ can be written uniquely as

$$u = u_{\mathcal{H}} + u_{\mathcal{V}},$$

with $u_{\mathcal{H}} \in \mathcal{H}_p$, called the *horizontal component* of u, and $u_{\mathcal{V}} \in \mathcal{V}_p$, called the *vertical component* of u; see Figure 15.2.

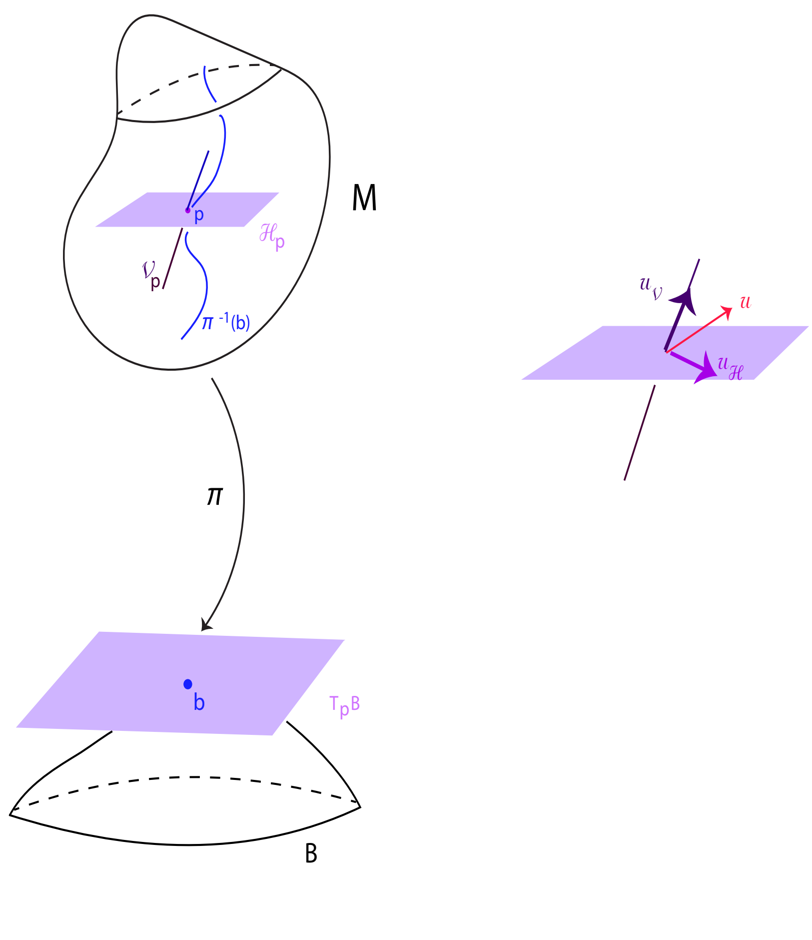

Figure 15.2: An illustration of a Riemannian submersion. Note $\mathcal{H}_p$ is isomorphic to T_bB.

Because π is a submersion, $d\pi_p$ gives a linear isomorphism between $\mathcal{H}_p$ and $T_{\pi(p)}B$.

If $d\pi_p$ is an isometry, then most of the differential geometry of B can be studied by "lifting" from B to M.

Definition 15.2. A map $\pi\colon M \to B$ between two Riemannian manifolds (M, g) and (B, h) is a *Riemannian submersion* if the following properties hold:

(1) The map π is a smooth submersion.

(2) For every $p \in M$, the map $d\pi_p$ is an isometry between the horizontal subspace $\mathcal{H}_p$ of T_pM and $T_{\pi(p)}B$.

We will see later that Riemannian submersions arise when B is a reductive homogeneous space, or when B is obtained from a free and proper action of a Lie group acting by isometries on B.

If $\pi\colon M \to B$ is a Riemannian submersion which is surjective onto B, then every vector field X on B has a unique *horizontal lift* $\overline{X}$ on M, defined such that for every $b \in B$ and every $p \in \pi^{-1}(b)$,

$$\overline{X}(p) = (d\pi_p)^{-1} X(b).$$

Since $d\pi_p$ is an isomorphism between $\mathcal{H}_p$ and $T_b B$, the above condition can be written

$$d\pi \circ \overline{X} = X \circ \pi,$$

which means that $\overline{X}$ and X are π-related (see Definition 8.5).

The following proposition is proved in O'Neill [38] (Chapter 7, Lemma 45) and Gallot, Hulin, Lafontaine [19] (Chapter 2, Proposition 2.109).

Proposition 15.8. *Let $\pi\colon M \to B$ be a Riemannian submersion between two Riemannian manifolds (M, g) and (B, h).*

(1) If γ is a geodesic in M such that $\gamma'(0)$ is a horizontal vector, then γ is horizontal geodesic in M (which means that $\gamma'(t)$ is a horizontal vector for all t), and $c = \pi \circ \gamma$ is a geodesic in B of the same length than γ. See Figure 15.3.

(2) For every $p \in M$, if c is a geodesic in B such that $c(0) = \pi(p)$, then for some ϵ small enough, there is a unique horizonal lift γ of the restriction of c to $[-\epsilon, \epsilon]$, and γ is a geodesic of M.

Furthermore, if $\pi\colon M \to B$ is surjective, then:

(3) For any two vector fields $X, Y \in \mathfrak{X}(B)$, we have

(a) $\langle \overline{X}, \overline{Y} \rangle = \langle X, Y \rangle \circ \pi$.

(b) $[\overline{X}, \overline{Y}]_{\mathcal{H}} = \overline{[X, Y]}$.

(c) $(\nabla_{\overline{X}} \overline{Y})_{\mathcal{H}} = \overline{\nabla_X Y}$, where ∇ is the Levi–Civita connection on M.

(4) If M is complete, then B is also complete.

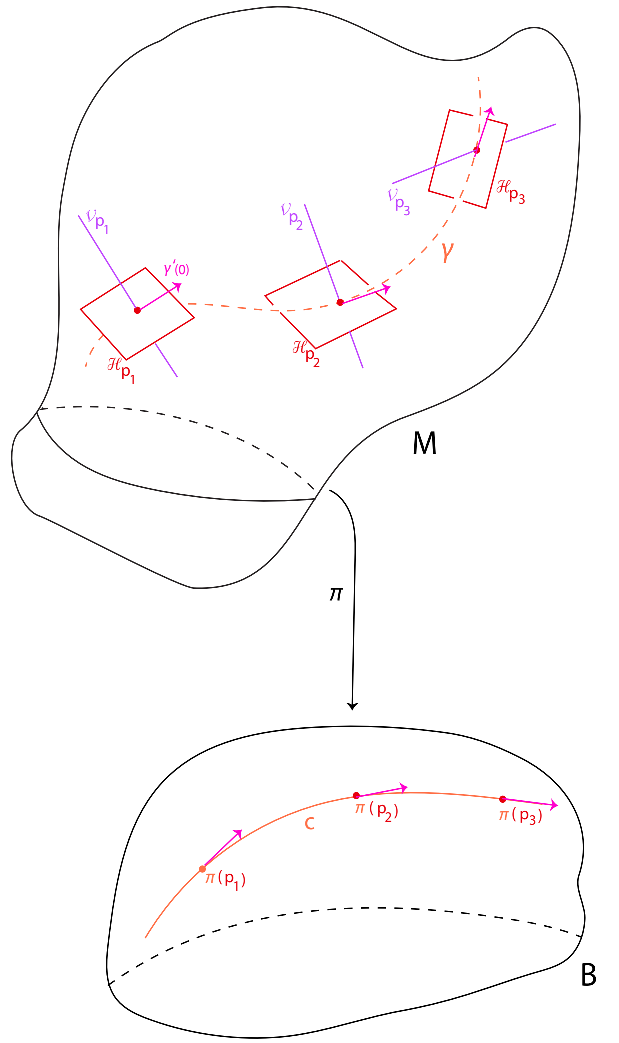

Figure 15.3: An illustration of Part (1), Proposition 15.8. Both γ and c are equal length geodesics in M and B respectively. All the tangent vectors to γ lie in horizontal subspaces.

In (2), we can't expect in general that the whole geodesic c in B can be lifted to M.

This is because the manifold (B, h) may be compete but (M, g) may not be. For example, consider the inclusion map $\pi\colon (\mathbb{R}^2 - \{0\}) \to \mathbb{R}^2$, with the canonical Euclidean metrics.

An example of a Riemannian submersion is $\pi\colon S^{2n+1} \to \mathbb{CP}^n$, where S^{2n+1} has the canonical metric and $\mathbb{CP}^n$ has the Fubini–Study metric.

Remark: It shown in Petersen [39] (Chapter 3, Section 5), that the connection $\nabla_{\overline{X}}\overline{Y}$ on M is given by

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]_{\mathcal{V}}.$$

15.4 Isometries and Killing Vector Fields ⊛

If X is a vector field on a manifold M, then we saw that we can define the notion of Lie derivative for vector fields $(L_X Y = [X, Y])$ and for functions $(L_X f = X(f))$.

It is possible to generalize the notion of Lie derivative to an arbitrary tensor field S (see Section **??**). In this section, we only need the following definition.

Definition 15.3. If $S = g$ (the metric tensor), then the *Lie derivative* $L_X g$ is defined by

$$L_X g(Y, Z) = X(\langle Y, Z\rangle) - \langle [X, Y], Z\rangle - \langle Y, [X, Z]\rangle,$$

with $X, Y, Z \in \mathfrak{X}(M)$, and where we write $\langle X, Y\rangle$ and $g(X, Y)$ interchangeably.

If Φ_t is an isometry (on its domain), then $\Phi_t^*(g) = g$, so $L_X g = 0$.

In fact, we have the following result proved in O'Neill [38] (Chapter 9, Proposition 23).

Proposition 15.9. *For any vector field X on a Riemannian manifold (M, g), the diffeomorphisms Φ_t induced by the flow Φ of X are isometries (on their domain) iff $L_X g = 0$.*

Informally, Proposition 15.9 says that $L_X g$ measures how much the vector field X changes the metric g.

Definition 15.4. Given a Riemannian manifold (M, g), a vector field X is a *Killing vector field* iff the Lie derivative of the metric vanishes; that is, $L_X g = 0$.

Killing vector fields play an important role in the study of reductive homogeneous spaces; see Section 20.4. They also interact with the Ricci curvature and play a crucial role in the Bochner technique; see Petersen [39] (Chapter 7).

As the notion of Lie derivative, the notion of covariant derivative $\nabla_X Y$ of a vector field Y in the direction X can be generalized to tensor fields (see Section **??**, and Proposition **??**).

In this section, we only need the following definition.

Definition 15.5. The *covariant derivative $\nabla_X g$ of the Riemannian metric g* on a manifold M is given by

$$\nabla_X(g)(Y, Z) = X(\langle Y, Z\rangle) - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Then observe that the connection ∇ on M is compatible with g iff $\nabla_X(g) = 0$ for all X.

Definition 15.6. We define the *covariant derivative ∇X of a vector field X* as the $(1, 1)$-tensor defined so that

$$(\nabla X)(Y) = \nabla_Y X$$

for all $X, Y \in \mathfrak{X}(M)$. For every $p \in M$, $(\nabla X)_p$ is defined so that $(\nabla X)_p(u) = \nabla_u X$ for all $u \in T_pM$.

The above facts imply the following Proposition.

Proposition 15.10. *Let (M, g) be a Riemannian manifold and let ∇ be the Levi–Civita connection on M induced by g. For every vector field X on M, the following conditions are equivalent:*

(1) X is a Killing vector field; that is, $L_X g = 0$.

(2) $X(\langle Y, Z\rangle) = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle$ for all $Y, Z \in \mathfrak{X}(M)$.

(3) $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all $Y, Z \in \mathfrak{X}(M)$; that is, ∇X is skew-adjoint relative to g.

Condition (3) shows that any parallel vector field is a Killing vector field.

Remark: It can be shown that if γ is any geodesic in M, then the restriction X_γ of X to γ is a Jacobi field (see Section 14.5), and that $\langle X, \gamma' \rangle$ is constant along γ (see O'Neill [38], Chapter 9, Lemma 26).