3.3 The L-Genus and the Todd Genus

The material in this section and the next two was first published in Hirzebruch [8].

Let B be a commutative ring with 1, and let Z, $\alpha_1, \ldots, \alpha_n, \ldots$ be some independent indeterminates, all of degree 1; make new independent indeterminates

$$q_j = \sigma_j(\alpha\text{'s}).$$

(The σ_j are the symmetric functions in the α's; for example, $q_1 = \alpha_1 + \cdots + \alpha_n$.) All computations are carried out in the ring $\mathcal{B} = B[[Z; \alpha_1, \ldots, \alpha_n, \ldots]]$. We have the subring $\mathcal{P} = B[[Z; q_1, \ldots, q_n, \ldots,]]$ and in $\mathcal{P}$, we have certain units (so-called *one-units*), namely

$$1 + \sum_{j \geq 1} b_j Z^j, \quad \text{where } b_j \in B.$$

If $Q(z)$ is a one-unit, $1 + \sum_{j\geq 1} b_j Z^j$, write

$$Q(z) = \prod_{j=1}^{\infty} (1 + \beta_j Z)$$

and call the β_j's the "roots" of Q. In the product $\prod_{l=1}^{\infty} Q(\alpha_j Z)$, the coefficient of Z^k is independent of the order of the α's and is a formal series in the elementary symmetric functions, q_j, of the α's. In fact, this coefficient has weight k and begins with $b_k q_1^k + \cdots$, call the coefficients $K_k^Q(q_1, q_2, \ldots, q_k)$. We deduce that

$$1 + \sum_{l=1}^{\infty} K_l^Q(q_1, q_2, \ldots, q_l) z^l = \prod_{l=1}^{\infty} Q(\alpha_j Z).$$

We see that a 1-unit, $Q(Z) = 1 + \sum_{j\geq 1} b_j Z^j$, yields a sequence of polynomials (in the elementary symmetric functions $q_1, \ldots, q_k$) of weights, $1, 2, \ldots$, say $\{K_l^Q\}_{l=1}^{\infty}$, called the *multiplicative sequence* of the 1-unit.

Conversely, given some sequence of polynomials, $\{K_l\}_{l=1}^{\infty}$, it defines an operator on 1-units to 1-units, call it K. Namely,

$$K(1 + \sum_{j\geq 1} q_j Z^j) = 1 + \sum_{l=1}^{\infty} K_l(q\text{'s}) Z^l.$$

So, Q gives the operator K^Q; namely,

$$K(1 + \sum_{j\geq 1} q_j Z^j) = 1 + \sum_{l=1}^{\infty} K_l^Q(q\text{'s}) Z^l.$$

Claim. When Q is given, the operator K^Q is multiplicative:

$$K^Q(1 + \sum_{j\geq 1} q_j' Z^j) K^Q(1 + \sum_{j\geq 1} q_j'' Z^j) = K^Q((1 + \sum_{j\geq 1} q_j' Z^j)(1 + \sum_{j\geq 1} q_j'' Z^j)).$$

Now, to see this, the left hand side is

$$[1 + \sum_{l=1}^{\infty} K_l^Q(q'\text{s}) Z^l][1 + \sum_{m=1}^{\infty} K_m^Q(q''\text{s}) Z^m] = \prod_{r=1}^{\infty} Q(\alpha_r' Z) \prod_{s=1}^{\infty} Q(\alpha_s'' Z) = \prod_{t=1}^{\infty} Q(\alpha_t Z),$$

where we have chosen some enumeration of the α's and the α''s, say $\alpha_1, \ldots, \alpha_t, \ldots = \alpha_1', \alpha_1'', \alpha_2', \alpha_2'', \ldots$. But,

$$\prod_{t=1}^{\infty} Q(\alpha_t Z) = 1 + \sum_{n=1}^{\infty} K_n^Q(\text{elem. symm. functions in } \alpha'\text{s and } \alpha''\text{s}) Z^n,$$

which is the right hand side of the assertion.

If conversely, we have some endomorphism of the 1-units under multiplication, say K, look at $K(1+Z) = 1 + \sum_{j \geq 1} a_j Z^j = Q(Z)$, some power series. Compute K^Q. We have

$$K^Q(1 + \sum_{j \geq 1} q_j Z^j) = \prod_{l=1}^{\infty} Q(\alpha_l Z),$$

where $1 + \sum_{j \geq 1} q_j Z^j = \prod_{j=1}^{\infty}(1 + \alpha_j Z)$. So, as K is multiplicative,

$$K(1 + \sum_{j \geq 1} q_j Z^j) = K(\prod_{j=1}^{\infty}(1 + \alpha_j Z)) = \prod_{j=1}^{\infty} K(1 + \alpha_j Z).$$

By definition of Q, the right hand side of the latter is

$$\prod_{l=1}^{\infty} Q(\alpha_l Z) = K^Q(1 + \sum_{j \geq 1} q_j Z^j)$$

and this proves:

Proposition 3.36 *The endomorphisms (under multiplication) of the* 1*-units are in one-to-one correspondence with the* 1*-units. The correspondence is*

$$\text{endo } K \rightsquigarrow 1\text{-}\textit{unit } K(1+Z),$$

and

$$1\text{-}\textit{unit } Q \rightsquigarrow \text{endo } K^Q.$$

We can repeat the above with new variables: X (for Z); c_j (for q_j); γ_j (for α_j); and connect with the above by the relations

$$Z = X^2; \alpha_l = \gamma_l^2.$$

This means

$$\sum_{i=0}^{\infty}(-1)^i q_i Z^i = \Big(\sum_{j=0}^{\infty} c_j X^j\Big)\Big(\sum_{r=0}^{\infty} c_r(-X)^r\Big) \qquad (*)$$

and if we set $\widetilde{Q}(X) = Q(X^2) = Q(Z)$, then

$$K_l^Q(q_1, \ldots, q_l) = K_{2l}^{\widetilde{Q}}(c_1, \ldots, c_{2l}) \quad \text{and} \quad K_{2l+1}^{\widetilde{Q}}(c_1, \ldots, c_{2l+1}) = 0.$$

For example, $(*)$ implies that $q_1 = c_1^2 - 2c_2$, etc.

Proposition 3.37 *If $B \supseteq \mathbb{Q}$, then there is one and only one power series, $L(Z)$, so that for all $k \geq 0$, the coefficient of Z^k in $L(Z)^{2k+1}$ is* 1*. In fact,*

$$L(Z) = \frac{\sqrt{Z}}{\tanh \sqrt{Z}} = 1 + \sum_{l=1}^{\infty}(-1)^{l-1} \frac{2^{2l}}{(2l)!} B_l Z^l.$$

Proof. For $k = 0$, we see that $L(Z)$ must be a 1-unit, $L(Z) = 1 + \sum_{j=1}^{\infty} b_j Z^j$. Consider $k = 1$; then, $L(Z)^3 = (1 + b_1 Z + O(Z^2))^3$, so

$$(1 + b_1 Z)^3 + O(Z^2) = 1 + 3b_1 Z + O(Z^2),$$

which implies $b_1 = 1/3$. Now, try for b_2: We must have

$$\begin{aligned}\left(1 + \frac{1}{3}Z + b_2 Z + O(Z^3)\right)^5 &= \left(1 + \frac{1}{3}Z + b_2 Z\right)^5 + O(Z^3)\\ &= \left(1 + \frac{1}{3}Z\right)^5 + 5\left(1 + \frac{1}{3}Z\right)^4 b_2 Z + O(Z^3)\\ &= \text{junk} + \left(\frac{10}{9} + 5b_2\right) Z^2 + O(Z^3).\end{aligned}$$

Thus,

$$5b_2 = 1 - \frac{10}{9} = -\frac{1}{9},$$

i.e., $b_2 = -1/45$. It is clear that we can continue by induction and obtain the existence and uniqueness of the power series.

Now, let

$$M(Z) = \frac{\sqrt{Z}}{\tanh\sqrt{Z}}.$$

Then, $M(Z)^{2k+1}$ is a power series and the coefficient of Z^k is (by Cauchy)

$$\frac{1}{2\pi i}\int_{|Z|=\epsilon} \frac{M(Z)^{2k+1}}{Z^{k+1}} dZ.$$

Let $t = \tanh\sqrt{Z}$. Then,

$$dt = \text{sech}^2\sqrt{Z}\left(\frac{1}{2\sqrt{Z}}\right) dZ,$$

so

$$\frac{M(Z)^{2k+1}}{z^{k+1}} dZ = \frac{\sqrt{Z}2\sqrt{Z}dt}{t^{2k+1}Z\text{sech}^2\sqrt{Z}} = \frac{2dt}{t^{2k+1}\text{sech}^2\sqrt{Z}}.$$

However, $\text{sech}^2 Z = 1 - \tanh^2 Z = 1 - t^2$, so

$$\frac{M(Z)^{2k+1}}{z^{k+1}} dZ = \frac{2dt}{t^{2k+1}(1 - t^2)}.$$

When t goes once around the circle $|t| = \text{small}(\epsilon)$, Z goes around twice around, so

$$\frac{1}{2\pi i}\int_{|t|=\text{small}(\epsilon)} \frac{2dt}{t^{2k+1}(1 - t^2)} = \text{twice what we want}$$

and our answer is

$$\frac{1}{2\pi i}\int_{|t|=\text{small}(\epsilon)} \frac{dt}{t^{2k+1}(1 - t^2)} = \frac{1}{2\pi i}\int_{|t|=\text{small}(\epsilon)} \frac{t^{2k}dt}{t^{2k+1}(1 - t^2)} + \text{other zero terms} = 1,$$

as required. ☐

Recall that

$$L(Z) = 1 + \frac{1}{3}Z - \frac{1}{45}Z^2 + O(Z^3).$$

Let us find $L_1(q_1)$ and $L_2(q_1, q_2)$. We have

$$\begin{aligned} 1 + L_1(q_1)Z + L_2(q_1,q_2)Z^2 + \cdots &= L(\alpha_1 Z)L(\alpha_2 Z) \\ &= \left(1 + \frac{1}{3}\alpha_1 Z - \frac{1}{45}\alpha_1^2 Z^2 + \cdots\right)\left(1 + \frac{1}{3}\alpha_2 Z - \frac{1}{45}\alpha_2^2 Z^2 + \cdots\right) \\ &= 1 + \frac{1}{3}(\alpha_1 + \alpha_2)Z + - \left(\frac{1}{45}(\alpha_1^2 + \alpha_2^2) + \frac{1}{9}\alpha_1\alpha_2\right) Z^2 + O(Z^3). \end{aligned}$$

We deduce that

$$L_1(q_1) = \frac{1}{3}q_1$$

and since $\alpha_1^2 + \alpha_2^2 = (\alpha_1 + \alpha_2)^2 - 2\alpha_1\alpha_2 = q_1^2 - 2q_2$, we get

$$L_2(q_1, q_2) = -\frac{1}{45}(7q_2 - q_1^2) = -\frac{1}{3^2 \cdot 5}(7q_2 - q_1^2).$$

Here are some more L-polynomials:

$$\begin{aligned} L_3 &= \frac{1}{3^3 \cdot 5 \cdot 7}(62q_3 - 13q_1q_2 + 2q_1^3) \\ L_4 &= \frac{1}{3^4 \cdot 5^2 \cdot 7}(381q_4 - 71q_3q_1 - 19q_2^2 + 22q_2q_1^2 - 3q_1^4) \\ L_5 &= \frac{1}{3^5 \cdot 5^2 \cdot 7 \cdot 11}(5110q_5 - 919q_4q_1 - 336q_3q_2 + 237q_3q_1^2 + 127q_2^2q_1 - 83q_2q_1^3 + 10q_1^5). \end{aligned}$$

Geometric application: Let X be an oriented manifold and let T_X be its tangent bundle. Take a multiplicative sequence, $\{K_l\}$, in the Pontrjagin classes of T_X: $p_1, p_2, \ldots$.

Definition 3.3 The *K-genus* (or *K-Pontrjagin genus*) of X is

$$\begin{cases} 0 & \text{if } \dim_{\mathbb{R}} X \not\equiv 0 \pmod 4, \\ K_n(p_1, \ldots, p_n)[X] & \text{if } \dim_{\mathbb{R}} X = 4n. \end{cases}$$

(a $4n$ rational cohomology class applied to a $4n$ integral homology class gives a rational number). When $K_l = L_l$ (our unique power series, $L(Z)$), we get the *L-genus* of X, denoted $L[X]$.

Look at $\mathbb{P}^{2n}_{\mathbb{C}}$, of course, we mean its tangent bundle, to compute characteristic classes. Write temporarily

$$\Theta = T_{\mathbb{P}^{2n}_{\mathbb{C}}}$$

a $\mathrm{U}(2n)$-bundle. We make $\zeta(\Theta)$ (remember, $\zeta\colon \mathrm{U}(2n) \to \mathrm{O}(4n)$), then we know

$$\sum_i p_i(\zeta(\Theta))(-Z)^i = \Big(\sum_j c_j(\Theta)X^j\Big)\Big(\sum_k c_k(\Theta)(-X)^k\Big),$$

with $Z = X^2$. Now, for projective space, $\mathbb{P}^{2n}_{\mathbb{C}}$,

$$1 + c_1(\Theta)t + \cdots + c_{2n}(\Theta)t^{2n} + t^{2n+1} = (1+t)^{2n+1}.$$

Therefore,

$$\sum_{i=0}^{2n} p_i(\zeta(\Theta))(-X^2)^i + \text{terms in } X^{4n+1}, X^{4n+2} = (1+X)^{2n+1}(1-X)^{2n+1} = (1-X^2)^{2n+1}.$$

Hence, we get

$$p_i(\zeta(\Theta)) = \binom{2n+1}{i} H^{2i}, \quad 1 \leq i \leq n.$$

Let K^L be the multiplicative homomorphism coming from the 1-unit, L. Then

$$\begin{aligned} K^L(1 + \sum_i p_i(-X^2)^i) &= \sum_j L_l(p_1, \ldots, p_l)(-X^2)^l \\ &= K^L((1 - X^2)^{2n+1}) \\ &= K^L(1 - X^2)^{2n+1} \\ &= L(-X^2)^{2n+1} = L(-Z)^{2n+1}. \end{aligned}$$

The coefficient of Z^n in the latter is $(-1)^n$ and by the first equation, it is $(-1)^n L_n(p_1, \ldots, p_n)$. Therefore, we have

$$L_n(p_1, \ldots, p_n) = 1, \quad \text{for every } n \geq 1.$$

Thus, we've proved

Proposition 3.38 *On the sequence of real 4n-manifolds: $\mathbb{P}^{2n}_{\mathbb{C}}$, $n = 1, 2, \ldots$, the L-genus of each, namely $L_n(p_1, \ldots, p_n)$, is 1. The L-genus is the unique genus having this property. Alternate form: If we substitute $p_j = \binom{2n+1}{j}$ in the L-polynomials, we get*

$$L_n\left(\binom{2n+1}{1}, \ldots, \binom{2n+1}{n}\right) = 1.$$

Now, for the Todd genus.

Proposition 3.39 *If $B \supseteq \mathbb{Q}$, then there is one and only one power series, $T(X)$, having the property: For all $k \geq 0$, the coefficient of X^k in $T(X)^{k+1}$ is 1. In fact this power series defines the holomorphic function*

$$\frac{X}{1 - e^{-X}}.$$

Proof. It is the usual induction, but we'll compute the first few terms. We see that $k = 0$ implies that T is a 1-unit, ie.,

$$T(X) = 1 + b_1 X + b_2 X^2 + O(X^3).$$

For $k = 1$, we have

$$T(X)^2 = (1 + b_1 X)^2 + O(X^2) = 1 + 2b_1 X + O(X^2),$$

so

$$b_1 = \frac{1}{2}.$$

For $k = 2$, we have

$$\begin{aligned} T(X)^3 &= \left(1 + \frac{1}{2}X + b_2 X^2\right)^3 + O(X^3) \\ &= \left(1 + \frac{1}{2}X\right)^3 + 3\left(1 + \frac{1}{2}X\right)^2 b_2 X^2 + O(X^3) \\ &= \text{stuff} + \frac{3}{4}X^2 + 3b_2 X^2 + O(X^3). \end{aligned}$$

Therefore, we must have

$$\frac{3}{4} + 3b_2 = 1,$$

that is,

$$b_2 = \frac{1}{12}.$$

So,

$$T(X) = 1 + \frac{1}{2}X + \frac{1}{12}X^2 + \cdots .$$

That

$$T(X) = \frac{X}{1 - e^{-X}}$$

comes from Cauchy's formula. □

From $T(X)$, we make the opertor K^T, namely,

$$K^T(1 + c_1X + c_2X^2 + \cdots) = 1 + \sum_{j=1}^{\infty} T_j(c_1, \ldots, c_j)X^j = \prod_{i=0}^{\infty} T(\gamma_i X),$$

where

$$1 + c_1X + c_2X^2 + \cdots = \prod_{i=0}^{\infty}(1 + \gamma_i X).$$

Let's work out $T_1(c_1)$ ans $T_2(c_1, c_2)$. From

$$1 + c_1X + c_2X^2 = (1 + \gamma_1X)(1 + \gamma_2X),$$

we get

$$\begin{aligned}
1 + T(c_1)X + T_2(c_1, c_2)X^2 + \cdots &= T(\gamma_1X)T(\gamma_2X) \\
&= \left(1 + \frac{1}{2}\gamma_1X + \frac{1}{12}\gamma_1^2X^2 + \cdots\right)\left(1 + \frac{1}{2}\gamma_2X + \frac{1}{12}\gamma_2^2X^2 + \cdots\right) \\
&= 1 + \frac{1}{2}(\gamma_1 + \gamma_2) + \left(\frac{1}{12}(\gamma_1^2 + \gamma_2^2) + \frac{1}{4}\gamma_1\gamma_2\right)X^2 + \cdots .
\end{aligned}$$

We get

$$T_1(c_1) = \frac{1}{2}c_1$$

and

$$T_2(c_1, c_2) = \frac{1}{12}(\gamma_1^2 + \gamma_2^2) + \frac{1}{4}\gamma_1\gamma_2 = \frac{1}{12}(c_1^2 - 2c_2) + \frac{1}{4}c_2 = \frac{1}{12}(c_1^2 + c_2).$$

i.e.,

$$T_2(c_1, c_2) = \frac{1}{12}(c_1^2 + c_2).$$

From this T, we make for a complex manifold, X, its *Todd genus*,

$$T_n(X) = T_n(c_1, \ldots, c_n)[X],$$

where $c_1, \ldots, c_n$ = Chern classes of T_X (the holomorphic tangent bundle) and $[X]$ = the fundamental homology class on $H_{2n}(X, \mathbb{Z})$. This is a rational number.

Suppose X and Y are two real oriented manifolds of dimensions n and r. Then

$$T_{X \prod Y} = pr_1^* T_X \amalg pr_2^* T_Y.$$

So, we have

$$1 + p_1(X \prod Y)Z + \cdots = pr_1^*(1 + p_1(X)Z + \cdots)pr_2^*(1 + p_1(Y)Z + \cdots). \qquad (\dagger)$$

Further observe that if ξ, η are cohomology classes for X, resp. Y, then $\xi \otimes 1$, $1 \otimes \eta$ are $pr_1^*(\xi)$, $pr_2^*(\eta)$, by Künneth and we have

$$\xi \otimes \eta[X \prod Y] = \xi[X]\eta[Y]. \tag{††}$$

Now, say K is an endomorphism of the 1-units from a given 1-unit, so it gives the K-genera of $X \prod Y$, X, Y. We have

$$\begin{aligned} K(1 + p_1(X \prod Y)Z + \cdots) &= K((1 + p_1(X)Z + \cdots)(1 + p_1(Y)Z + \cdots)) \\ &= K(1 + p_1(X)Z + \cdots)K(1 + p_1(Y)Z + \cdots). \end{aligned}$$

Now, evaluate on $[X \prod Y]$, find a cycle of $X \prod Y]$ in $H_{n+r}(X \prod Y, \mathbb{Z})$. By (††), we get

$$K_{n+r}(p_1, \ldots, p_{n+r})[X \prod Y] = K_n(p_1, \ldots, p_n)[X]K_r(p_1, \ldots, p_r)[Y]$$

and

Proposition 3.40 *If K is an endomorphism of 1-units, then the K-genus is multiplicative, i.e.,*

$$K(X \prod Y) = K(X)K(Y).$$

Interpolation among the genera (of interest).

Let y be a new variable (the interpolation variable). Make a new function, with coefficients in $B \supseteq \mathbb{Q}[y]$,

$$Q(y; x) = \frac{x(y+1)}{1 - e^{-x(y+1)}} - xy$$

(First form of $Q(y; x)$). We can also write

$$\begin{aligned} Q(y; x) &= \frac{x(y+1)e^{x(y+1)}}{e^{x(y+1)} - 1} - xy \\ &= \frac{x(y+1)(e^{x(y+1)} - 1 + 1)}{e^{x(y+1)} - 1} - xy \\ &= x(y+1) + \frac{x(y+1)}{e^{x(y+1)} - 1} - xy \\ &= \frac{x(y+1)}{e^{x(y+1)} - 1} + x. \end{aligned}$$

(Second form of $Q(y; x)$).

Let us compute the first three terms of $Q(y; x)$. As

$$e^{-x(y+1)} = 1 - x(y+1) + \frac{(x(y+1))^2}{2!} + \cdots + (-1)^k \frac{(x(y+1))^k}{k!} + \cdots,$$

we have

$$1 - e^{-x(y+1)} = x(y+1) - \frac{(x(y+1))^2}{2!} + \cdots + (-1)^{k-1} \frac{(x(y+1))^k}{k!} + \cdots$$

and so,

$$\frac{x(y+1)}{1 - e^{-x(y+1)}} = \left[1 + \cdots + (-1)^{k-1} \frac{(x(y+1))^{k-1}}{k!} + \cdots\right]^{-1}.$$

If we denote this power series by $1+\alpha_1x+\alpha_2x^2+\cdots$, we can solve for α_1, α_2, etc., by solving the equation

$$1 = (1+\alpha_1x+\alpha_2x^2+\cdots)\left[1-\frac{x(y+1)}{2}+\cdots+(-1)^{k-1}\frac{(x(y+1))^{k-1}}{k!}+\cdots\right].$$

This implies

$$\alpha_1 = \frac{(y+1)}{2}$$

and

$$\alpha_2 = \frac{1}{4}(y+1)^2 - \frac{1}{6}(y+1)^2 = \frac{1}{12}(y+1)^2.$$

Consequently,

$$Q(y;x) = 1+\frac{x(y+1)}{2}+\frac{1}{12}x^2(y+1)^2+O(x^3(y+1)^3)-xy,$$

i.e.,

$$Q(y;x) = 1+\frac{x(1-y)}{2}+\frac{1}{12}x^2(y+1)^2+O(x^3(y+1)^3).$$

Make the corresponding endomorphisms, $\mathcal{T}_y$. Recall,

$$\mathcal{T}_y(1+c_1X+\cdots+c_nX^n+\cdots) = \begin{cases}\prod_{j=1}^{\infty} Q(y;\gamma_jX)\\ \sum_{j=0}^{\infty} T_j(y;c_1,\ldots,c_j)X^j,\end{cases}$$

where, of course,

$$1+c_1X+\cdots+c_nX^n+\cdots = \prod_{j=1}^{\infty}(1+\gamma_iX).$$

We obtain the $\mathcal{T}_y$-genus. The 1-unit, $Q(y;x)$, satisfies

Proposition 3.41 *If $B \supseteq \mathbb{Q}[y]$, then there exists one and only one power series (it is our $Q(y;x)$) in $B[[x]]$ (actually, $\mathbb{Q}[y][[x]]$) so that, for all $k \geq 0$, the coefficient of X^k in $Q(y;x)^{k+1}$ is $\sum_{i=0}^{k}(-1)^iy^i$.*

Proof. The usual (by induction). Let us check for $k=1$. We have

$$Q(y;x)^2 = \left(1+\frac{x(1-y)}{2}\right)^2 + O(x^2) = 1+(1-y)x+O(x^2).$$

The coefficient of x is indeed $1-y=\sum_{i=0}^{1}(-1)^iy^i$. $\square$

Look at $Q(y;x)$ for $y=1,-1,0$. Start with -1. We have

$$Q(-1;x) = 1+x.$$

Now, for $y=0$, we get

$$Q(0;y) = T(X) = \frac{x}{1-e^{-x}}.$$

Finally, consider $y=1$. We have

$$\begin{aligned} Q(1;x) &= \left(\frac{2}{1-e^{-2x}}-1\right)x\\ &= \left(\frac{2e^{2x}}{e^{2x}-1}-1\right)x\\ &= \left(\frac{e^{2x}+1}{e^{2x}-1}\right)x\\ &= \frac{x}{\tanh x} = L(x^2).\end{aligned}$$

We proved $Q(y;x)$ is the unique power series in $Q[y][[x]]$ so that the coefficient of x^k in $Q(y;x)^{k+1}$ is $\sum_{i=0}^{k}(-1)^i y^i$. Therefore, we know (once again) that $Q(0;x) = Q(x) =$ the unique power series in $\mathbb{Q}[x]$ so that the coefficient of x^k in $Q(x)^{k+1}$ is 1. Since, for projective space, $\mathbb{P}^k_{\mathbb{C}}$, we have

$$1 + c_1 X + \cdots + c_k X^K + X^{k+1} = (1+X)^{k+1}$$

and since

$$K^Q((1+X)^{k+1}) = \begin{cases} K^Q(1+X)^{k+1} = Q(X)^{k+1} \\ \sum_{l=0}^{\infty} T_l(c_1, \cdots, c_l) X^l \end{cases}$$

we get

$$T_k(c_1, \ldots, c_k) = 1$$

when the c's come from $\mathbb{P}^k_{\mathbb{C}}$ and if $T_k(y; c_1, \ldots, c_k)$ means the corresponding object for $Q(y;x)$, we get

Proposition 3.42 *The Todd genus, $T_n(c_1, \ldots, c_n)$, and the T_y-genus, $T_n(y; c_1, \ldots, c_n)$, are the only genera so that on* **all** *$\mathbb{P}^n_{\mathbb{C}}$ ($n = 0, 1, 2, \ldots$) they have values 1, resp. $\sum_{i=0}^{\infty}(-1)^i y^i$.*

Write $\mathcal{T}_y$ for the multiplicative operator obtained from $Q(y;x)$, i.e.,

$$\mathcal{T}_y(1 + c_1 X + \cdots + c_j X^j + \cdots) = \sum_{n=0}^{\infty} T_n(y; c_1, \ldots, c_n) X^n.$$

Equivalently,

$$\mathcal{T}_y(1 + c_1 X + \cdots + c_j X^j + \cdots) = \prod_{j=1}^{\infty} Q(y; \gamma_j X),$$

where

$$(1 + c_1 X + \cdots + c_j X^j + \cdots = \prod_{j=1}^{\infty} (1 + \gamma_j X).$$

Now, for all n, the expression $T_n(y; c_1, \ldots, c_n)$ is some polynomial (with coefficients in the c's) of degree at most n in y. Thus, we can write

$$T_n(y; c_1, \ldots, c_n) = \sum_{l=0}^{n} T_n^{(l)}(y; c_1, \ldots, c_n) y^l,$$

and this is new polynomial invariants, the $T_n^{(l)}(y; c_1, \ldots, c_n)$.

We have

$$T_n(-1; c_1, \ldots, c_n) = \sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = c_n,$$

by the fact that $Q(-1; x) = 1 + x$. Next, when $y = 0$,

$$T_n(0; c_1, \ldots, c_n) = T_n^{(0)}(c_1, \ldots, c_n) = T_n(c_1, \ldots, c_n).$$

When $y = 1$, then

$$T_n(1; c_1, \ldots, c_n) = \sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ L_{\frac{n}{2}}(p_1, \ldots, p_{\frac{n}{2}}) & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we get

Proposition 3.43 *If $B \supseteq \mathbb{Q}[y]$, then we have*

(A) $\sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = c_n$, *for all* n.

(B) $T_n^{(0)}(c_1, \ldots, c_n) = \mathrm{td}(c_1, \ldots, c_n)\ (= T_n(c_1, \ldots, c_n))$.

(C)

$$\sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ L_{\frac{n}{2}}(p_1, \ldots, p_{\frac{n}{2}}) & \text{if } n \text{ is even.} \end{cases}$$

The *total Todd class* of a vector bundle, ξ, is

$$\mathrm{td}(\xi)(t) = \sum_{j=0}^{\infty} \mathrm{td}_j(c_1, \ldots, c_j)t^j = 1 + \frac{1}{2}c_1(\xi) + \frac{1}{12}(c_1^2(\xi) + c_2(\xi))t^2 + \frac{1}{24}(c_1(\xi)c_2(\xi))t^3 + \cdots.$$

Here some more Todd polynomials:

$$\begin{aligned} T_4 &= \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4) \\ T_5 &= \frac{1}{1440}(-c_4c_1 + c_3c_1^2 + 3c_2^2c_1 - c_2c_1^3) \\ T_6 &= \frac{1}{60480}(2c_6 - 2c_5c_1 - 9c_4c_2 - 5c_4c_1^2 - c_3^3 + 11c_3c_2c_1 + 5c_3c_1^3 + 10c_2^3 + 11c_2^2c_1^2 - 12c_2c_1^4 + 2c_1^6). \end{aligned}$$

Say

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$

is an exact sequence of vector bundles. Now,

$$(1 + c_1't + \cdots + c_{q'}'t^{q'})(1 + c_1''t + \cdots + c_{q''}''t^{q''}) = 1 + c_1t + \cdots + c_qt^q,$$

and td is a multiplicative sequence, so

$$\mathrm{td}(\xi')(t)\mathrm{td}(\xi'')(t) = \mathrm{td}(\xi)(t).$$

Let us define the *K-ring* of vector bundles. As a group, this is the free abelian group of isomorphism classes of vector bundles modulo the equivalence relation

$$[V] = [V'] + [V'']$$

iff

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 \quad \text{is exact.}$$

For the product, define

$$[V] \cdot [W] = [V \otimes W].$$

The ring K is a graded ring by rank (the rank of the vb).

Say ξ is a vector bundle and

$$1 + c_1t + \cdots + c_qt^q + \cdots = \prod(1 + \gamma_jt).$$

Remember,

$$1 + \mathrm{td}_1(c_1)t + \cdots + \mathrm{td}_n(c_1, \ldots, c_n)t^n + \cdots = \prod T(\gamma_jt) = \prod \frac{\gamma_jt}{1 - e^{-\gamma_jt}}.$$

Now, we define the *Chern character* of a vector bundle. For

$$1 + c_1 t + \cdots + c_q t^q + \cdots = \prod (1 + \gamma_j t)$$

set

$$\mathrm{ch}(\xi)(t) = \sum_j e^{\gamma_j t} = \mathrm{ch}_0(\xi) + \mathrm{ch}_1(\xi)t + \cdots + \mathrm{ch}_n(\xi)t^n + \cdots,$$

where $\mathrm{ch}_j(\xi)$ is a polynomial in $c_1, \ldots, c_j$ of weight j. Since

$$e^{\gamma_j t} = \sum_{r=0}^{\infty} \frac{(\gamma_j t)^r}{r!},$$

we have

$$\sum_j e^{\gamma_j t} = \sum_j \sum_r e^{\gamma_j t} = \sum_r \Big(\frac{1}{r!} \sum_j \gamma_j^r\Big) t^r,$$

which shows that

$$\mathrm{ch}_r(c_1, \ldots, c_r) = \frac{1}{r!} \sum_j \gamma_j^r = \frac{1}{r!} s_r(\gamma_1, \ldots, \gamma_q).$$

The sums, s_r, can be computed by induction using Newton's formulae:

$$s_l - s_{l-1}c_1 + s_{l-2}c_2 + \cdots + (-1)^{l-1} s_1 c_{l-1} + (-1)^l c_l = 0.$$

(Recall, $c_j = \sigma_j(\gamma_1, \ldots, \gamma_q)$.) We have

$$\begin{aligned}
\mathrm{ch}_1(c_1) &= c_1 \\
\mathrm{ch}_2(c_1, c_2) &= \frac{1}{2}(c_1^2 - 2c_2) \\
\mathrm{ch}_3(c_1, c_2, c_3) &= \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\
\mathrm{ch}_4(c_1, c_2, c_3, c_4) &= \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4).
\end{aligned}$$

Say

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$

is an exact sequence of bundles. The Chern roots of ξ are the Chern roots of ξ' together with those of ξ''. The definition implies

$$\mathrm{ch}(\xi)(t) = \mathrm{ch}(\xi')(t) + \mathrm{ch}(\xi'')(t).$$

If ξ and η are vector bundles with Chern roots, $\gamma_1, \ldots, \gamma_q$ and $\delta_1, \ldots, \delta_r$, then $\xi \otimes \eta$ has Chern roots $\gamma_i + \delta_j$, for all i, j. By definition,

$$\mathrm{ch}(\xi \otimes \eta)(t) = \sum_{i,j} e^{(\gamma_j + \delta_j)t} = \sum_{i,j} e^{\gamma_j t} e^{\delta_j t} = \Big(\sum_i e^{\gamma_j t}\Big)\Big(\sum_j e^{\gamma_j t}\Big) = \mathrm{ch}(\xi)(t)\mathrm{ch}(\eta)(t).$$

The above facts can be summarized in the following proposition:

Proposition 3.44 *The Chern character,* $\mathrm{ch}(\xi)(t)$*, is a ring homomorphism from* $K(\mathrm{vector}(X))$ *to* $H^*(X, \mathbb{Q})$.

If ξ is a $\mathrm{U}(q)$-vector bundle over a complex analytic manifold, X, write

$$T(X,\xi)(t) = \mathrm{ch}(\xi)(t)\mathrm{td}(\xi)(t),$$

the *T-characteristic* of ξ over X.

Remark: The $T_n^{(l)}$ satisfy the *duality formula*

$$(-1)^n T_n^{(l)}(c_1,\ldots,c_n) = T_n^{(n-l)}(c_1,\ldots,c_n).$$

To compute them, we can use

$$T_n^{(l)}(c_1,\ldots,c_n) = \kappa_n(\mathrm{ch}(\bigwedge^l \xi^D)(t)\mathrm{td}(\xi)(t)),$$

where $c_1,\ldots,c_n$ are the Chern classes of the v.b., ξ, and κ_n always means the term of total degree n.

3.4 Cobordism and the Signature Theorem

Let M be a real oriented manifold. Now, if $\dim(M) \equiv 0\,(4)$, we have the Pontrjagin classes of M, say $p_1, \ldots, p_n$ (with $\dim(M) = 4n$). Say $j_1 + \cdots + j_r = n$ (a partition of n) and let $\mathcal{P}(n)$ denote all partitions of n. Write this as (j). Consider $p_{j_1} \cdots p_{j_r}$, the product of weight $j_1, \ldots, j_r$ monomials in the p's; this is in $H^{4n}(M, \mathbb{Z})$. Apply $p_{j_1} \cdots p_{j_r}$ to $[M]$ = fundamental cycle, we get an integer. Such an integer is a *Pontrjagin number* of M, there are $\#(\mathcal{P}(n))$ of them.

Since
$$(-1)^i p_i Z^i = \Big(\sum c_j X^j\Big)\Big(\sum c_l (-X)^l\Big),$$
the Pontrjagin classes are independent of the orientation. introduce $-M$, the manifold M with the opposite orientation. Then,
$$p_{j_1} \cdots p_{j_r}[-M] = -p_{j_1} \cdots p_{j_r}[M].$$

Define the *sum*, $M + N$, of two manifolds M and N as $M \amalg N$, their disjoint union, again, oriented. We have
$$H^*(M + N, \mathbb{Z}) = H^*(M, \mathbb{Z}) \prod H^*(N, \mathbb{Z})$$
and consequently, the Pontrjagin numbers of $M + N$ are the sums of the Pontrjagin numbers of M and N.

We also define $M \prod N$, the cartesian product of M and N. By Künneth,
$$[M \prod N] = [M \otimes 1][1 \otimes N],$$
so the Pontrjagin numbers of $M \prod N$ are the products of the Pontrjagin numbers of M and N.

The Pontrjagin numbers of manifolds of dimension $n \not\equiv 0\,(4)$ are all zero.

We make an equivalence relation (*Pontrjagin equivalence*) on oriented manifolds by saying that
$$M \equiv N\,(P)$$
iff every Pontrjagin number of M is the equal to the corresponding Pontrjagin number of N. Let $\widetilde{\Omega}_n$ be the set of equivalence classes of dimension n manifolds, so that $\widetilde{\Omega}_n = (0)$ iff $n \not\equiv 0\,(4)$ and
$$\coprod_{n \geq 0} \widetilde{\Omega}_n = \coprod_{r \geq 0} \widetilde{\Omega}_{4r}.$$

We see that $\widetilde{\Omega}$ is a graded abelian torsion-free group. For $\widetilde{\Omega} \otimes_{\mathbb{Z}} \mathbb{Q}$, a ring of interest.

Proposition 3.45 *For a sequence, $\{M_{4k}\}_{k=0}^{\infty}$ of manifolds, the following are equivalent:*

(1) For every k, $s_k[M_{4k}] \neq 0$. Here, write $1 + p_1 X + \cdots + p_n Z^n$ as a product $\prod_{j=1}^{m \geq n}(1 + \beta_j Z)$, where equality means up to terms of degree n if $m > n$ and then
$$s_k = \beta_1^k + \cdots + \beta_m^k (m \geq k)$$
a polynomial in $p_1, \ldots, p_k$, of weight k, so it makes sense on M_{4k}.

(2) The mapping from multiplicative sequences with coefficients in B ($\supseteq \mathbb{Q}$) to $\prod_{\aleph_0} B$, via
$$\{K_j\}_{j=1}^{\infty} \mapsto (K_1[M_1], \ldots, (K_k[M_k], \ldots)$$
is a bijection. That is, given any sequence $a_1, \ldots, a_k, \ldots$ of elements of B, there is one and only one multiplicative sequence, $\{K_l\}$ (coeffs in B), so that
$$K_k(p_1, \ldots, p_k)[M_{4k}] = a_k.$$

Proof. (1) $\Longrightarrow$ (2). Choose $a_1, a_2, \ldots$ from B. Now, multiplicative sequences with coefficients in B are in one-to-one correspondence with one-units of $B[[z]]$, say $Q(z)$ is the 1-unit. If

$$1 + p_1 Z + \cdots + p_k Z^k + \cdots = \prod_j (1 + \beta_j Z),$$

then

$$1 + K_1(p_1)Z + \cdots + K_k(1, \ldots, p_k)Z^k + \cdots = \prod_j Q(\beta_j Z).$$

We must produce a unique 1-unit $1 + b_1 Z + \cdots = Q(Z)$, so that a_k is equal to the coefficient of Z^k applied to M_{4k} in $\prod_j Q(\beta_j Z) b_k +$ some polynomial in $b_1, \ldots, b_{k-1}$, of weight k. This polynomial has $\mathbb{Z}$-coefficients and depends on the M_{4k}. We need

$$a_k = s_k[M_{4k}] + \text{poly in } b_1, \ldots, b_{k-1} \tag{$\dagger$}$$

By (1), all $s_l[M_{4k}] \neq 0$; by induction we can find unique b_k's from the a_k's.

(2) $\Longrightarrow$ (1). By (2), the equations ($\dagger$) have a unique b-solution given the a's. But then, all $s_k[M_{4k}] \neq 0$, else no unique solution or worse, no solution. $\square$

Corollary 3.46 *The sequence $\{\mathbb{P}^{2k}_{\mathbb{C}}\}$ satisfies (1) and (2). Such a sequence is called a basis sequence for the n-manifolds.*

Proof. We have

$$1 + p_1 Z + \cdots + p_k Z^k = (1 + h^2 Z)^{2k+1},$$

where $h^2 \in H^4(\mathbb{P}^{2k}_{\mathbb{C}}, \mathbb{Z})$ (square of the hyperplane class). But then, $\beta_j = h^2$, for $j = 1, \ldots, 2k+1$ and

$$s_k(\mathbb{P}^{2k}_{\mathbb{C}}) = \sum_{j=1}^{2k+1} h^{2k}(\mathbb{P}^{2k}_{\mathbb{C}}) = 2k + 1 \neq 0$$

establishing the corollary. $\square$

Theorem 3.47 *Suppose $\{M_{4k}\}$ is a basis sequence for $\widetilde{\Omega} \otimes \mathbb{Q}$. Then, each $\alpha \in \widetilde{\Omega} \otimes \mathbb{Q}$ has the unique form $\sum_{(j)} \rho_{(j)} M_{(j)}$, where*

(1) $(j) = (j_1, \ldots, j_r)$; $j_1 + \cdots + j_r = k$; $M_{(j)} = M_{4j_1} \prod \cdots \prod M_{4j_r}$.

(2) $\rho_{(j)} \in \mathbb{Q}$. Secondly, given any rational numbers, $\rho_{(j)}$, there is some $\alpha \in \widetilde{\Omega} \otimes \mathbb{Q}$ so that

$$p_{(j)}(\alpha) = p_{j_1} p_{j_2} \cdots p_{j_r}(\alpha) = \rho_{(j)}.$$

(3) Given any sequence, $\{M_{4k}\}$, of manifolds suppose $\alpha = \sum_{(j)} \rho_{(j)} M_{(j)}$, then, for every $k \geq 0$, we have $s_k(\alpha) = \rho_k s_k(M_{4k})$.

(4) If each $\alpha \in \widetilde{\Omega} \otimes \mathbb{Q}$ is a sum $\sum_{(j)} \rho_{(j)} M_{(j)}$, then the $\{M_{4k}\}$ are a basis sequence. So, the $\{M_{4k}\}$ are a basis sequence iff the monomials $M_{(j)} = M_{4j_1} \prod \cdots \prod M_{4j_r}$ (over $\mathcal{P}(k)$, all k) form a basis of $\widetilde{\Omega} \otimes \mathbb{Q}$ in the usual sense.

Proof. Note that, as abelian group, $\widetilde{\Omega}_{4k}$ has rank $\#(\mathcal{P}(k))$ (the number of Pontrjagin numbers of weight k is $\#(\mathcal{P}(k))$).

(1) Pick indeterminates $q_1, \ldots, q_l$ over $\mathbb{Q}$ and choose any integer $l \geq 0$. By the previous proposition, since $\{M_{4k}\}$ is a basis sequence there exists one and only one multiplicative sequence, call it $\{K^{(l)}_m\}_{m=1}^{\infty}$, so that

$$K^{(l)}_m[M_{4m}] = q^l_m.$$

We need only check our conclusion for $\alpha \in \widetilde{\Omega}_{4k} \otimes \mathbb{Q}$ for fixed k. Now,

$$\dim_{\mathbb{Q}} \widetilde{\Omega}_{4k} \otimes \mathbb{Q} = \#(\mathcal{P}(k))$$

and there exist exactly $\#(\mathcal{P}(k))$ elements $M_{(j)}$, so all we need to show is

$$\sum_{(j)} \rho_{(j)} M_{(j)} = 0 \quad \text{implies all } \rho_{(j)} = 0.$$

Suppose $\sum_{(j)} \rho_{(j)} M_{(j)} = 0$ and apply the multiplicative sequence $\{K_m^{(l)}\}_{m=1}^{\infty}$. We get

$$\sum_{(j)} \rho_{(j)} q_{j_1}^l \cdots q_{j_r}^l = 0 \quad \text{for all } l \geq 0. \tag{$*$}$$

Write $q_{j_1}^l \cdots q_{j_r}^l = q_{(l)}^l$. The $q_{(l)}^l$ are all pairwise distinct, so by choosing enough l, the equation $(*)$ gives a system of linear equations (unknowns the $\rho_{(j)}$) with a Vandermonde determinant. By linear algebra, all $\rho_{(j)} = 0$.

(2) This is now clear as the $M_{(j)}$ span $\widetilde{\Omega}_{4k} \otimes \mathbb{Q}$ for all k.

(3) Look at $Q(Z) = 1 + Z^k$ and make the corresponding multiplicative sequence. We have

$$1 + k_1(p_1)Z + \cdots + K_k(p_1, \ldots, p_k)Z^k + \cdots = \prod_{j \geq k} (1 + \beta_j^k Z^k).$$

Therefore, $K_l(p_1, \ldots, p_l) = 0$ if $l < k$ and $K_k(p_1, \ldots, p_k) = \beta_1^k + \beta_2^k + \cdots = s_k$. Apply this multiplicative sequence to α, we get $s_k(\alpha) = \rho_k s_k(M_{4k})$, as required.

(4) Suppose each $\alpha = \sum_{(j)} \rho_{(j)} M_{(j)}$, yet, for some k, $s_k(M_k) = 0$. By (3), we have $s_k(\alpha) = \rho_k s_k(M_{4k}) = 0$. It follows that $s_l(\alpha) = 0$, for all α. Now, let $\alpha = \mathbb{P}_{\mathbb{C}}^{2k}$. We get

$$2k + 1 = s_k(\alpha) = 0,$$

a contradiction. □

Corollary 3.48 *The map $M_{4k} \mapsto Z_k$ (and $M_{(j)} \mapsto Z_{j_1} \cdots Z_{j_r}$) gives a $\mathbb{Q}$-algebra isomorphism $\widetilde{\Omega} \otimes \mathbb{Q} \cong \mathbb{Q}[Z_1, Z_2, \ldots]$, where $\deg(Z_l) = 4l$. (Here, $\{M_{4k}\}$ is a basis sequence.)*

Corollary 3.49 *The $\mathbb{Q}$-algebra maps, $\widetilde{\Omega} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$, are in one-to-one correspondence with the multiplicative sequences with coefficients in $\mathbb{Q}$ (or, what's the same, with the 1-units of $\mathbb{Q}[[Z]]$). The map is*

$$\alpha \in \widetilde{\Omega} \otimes \mathbb{Q} \mapsto K(\alpha).$$

Proof. Multiplicative sequences correspond to 1-units $1 + b_1 Z + \cdots +$ and (†) above shows we know the b's iff we know the value of the homomorphism on the M_{4k}, i.e., on the Z_k's and then, use Corollary 3.48. □

Note that manifolds with boundary also have a notion of orientation.

An oriented n-dimensional manifold, M, *bounds* iff there is an oriented manifold, V and an orientation preserving diffeomorphism, $\partial V \cong M$.

Definition 3.4 *(R. Thom)* Two manifolds, M and N are *cobordant* if $M + (-N)$ bounds.

Introduce *cobordism*, the equivalence relation

$$M \equiv N \ (C) \quad \text{iff } M \text{ is cobordant to } N.$$

We see immediately that if $M \equiv N \ (C)$ and $M' \equiv N' \ (C)$, then

(1) $M \amalg M' \equiv N \amalg N'$ (C)

(2) $-M \equiv -N$ (C)

(3) $M \prod M' \equiv N \prod N'$ (C).

Using this equivalence, we have the graded abelian group (under $\amalg$)

$$\Omega = \coprod_n \Omega_n,$$

where Ω_n is the set of equivalence classes of n-dimensional oriented manifolds under cobordism. We make Ω into a ring as follows: Given $\alpha \in \Omega_m$ and $\beta \in \Omega_n$, then

$$\alpha\beta = \text{class of}(\alpha \prod \beta)$$

and (use homology), $\alpha\beta = (-1)^{mn}\beta\alpha$. We call Ω the *oriented cobordism ring*.

Theorem 3.50 *(Pontrjagin) If M bounds (i.e., $M \equiv 0$ (C)) then* **all** *its Pontrjagin numbers vanish (i.e., $M \equiv 0$ (P)). Hence, there is a surjection $\Omega \longrightarrow \widetilde{\Omega}$ and hence a surjection $\Omega \otimes \mathbb{Q} \longrightarrow \widetilde{\Omega} \otimes \mathbb{Q}$.*

Proof. We have $M = \partial V$, write $i\colon M \hookrightarrow V$ for the inclusion. Let $p_1, \ldots, p_l, \ldots$ be the Pontrjagin classes of T_V; note, as $M = \partial V$,

$$i^* T_V = T_V \restriction \partial V = T_V \restriction M = T_M \amalg \mathbb{I},$$

where $\mathbb{I}$ denotes the trivial bundle. Therefore, the Pontrjagin classes of M are i^*(those of V). So, for $4k = \dim M$ and $j_1 + \cdots + j_r = k$,

$$p_{(j)}[M] = i^*((p_{j_1} \cdots p_{j_r})[M]),$$

where $[M]$ is the $4k$-cycle in $H_{4k}(V, \mathbb{Z})$. But, $[M] = 0$ in $H_{4k}(V, \mathbb{Z})$, as $M = \partial V$. Therefore, the right hand side is zero. $\square$

We will need a deep theorem of René Thom. The proof uses a lot of homotopy theory and is omitted.

Theorem 3.51 *(R. Thom, 1954, Commentari) The groups Ω_n of oriented n-manifolds are finite if $n \not\equiv 0 \,(\mathrm{mod}\, 4)$ and Ω_{4k} = free abelian group of rank $\#(\mathcal{P}(k)) \amalg$ finite abelian group. Hence, $\Omega_n \otimes \mathbb{Q} = (0)$ if $n \not\equiv 0 \,(\mathrm{mod}\, 4)$ and $\dim(\Omega_{4k} \otimes \mathbb{Q}) = \#(\mathcal{P}(k)) = \dim(\widetilde{\Omega}_{4k} \otimes \mathbb{Q})$. We conclude that the surjection $\Omega \otimes \mathbb{Q} \longrightarrow \widetilde{\Omega} \otimes \mathbb{Q}$ is an isomorphism. Therefore,*

$$\Omega \otimes \mathbb{Q} \cong_{\text{alg}} \mathbb{Q}[Z_1, \ldots, Z_n, \ldots].$$

We will also need another theorem of Thom. First, recall the notion of index of a manifold, from Section 2.6. The index of M, denoted $I(M)$ is by definition the signature, $\mathrm{sgn}(Q)$, where Q is the intersection form on the middle cohomology, $H^n(M, \mathbb{C})$, when n is even. So, $I(M)$ makes sense if $\dim_{\mathbb{R}} M \equiv 0\,(4)$.

Theorem 3.52 *(R. Thom, 1952, Ann. Math. ENS) If the n-dimensional oriented manifold bounds, then $I(M) = 0$.*

In view of these two theorems we can reformulate our algebraic theorem on $\mathrm{Hom}_{\mathbb{Q}\text{-alg}}(\widetilde{\Omega} \otimes \mathbb{Q}, \mathbb{Q})$ in terms of $\Omega \otimes \mathbb{Q}$.

Theorem 3.53 *Suppose λ is a function from oriented n-manifolds to $\mathbb{Q}$, $M \mapsto \lambda(M)$, satisfying*

(1) $\lambda(M + N) = \lambda(M) + \lambda(N)$; $\lambda(-M) = -\lambda(M)$.

(2) If M bounds, then $\lambda(M) = 0$.

(3) If $\{M_{4k}\}$ is a basis sequence for Ω, then when $j_1 + \cdots + j_r = k$, we have

$$\lambda\Big(M_{4j_1} \prod \cdots \prod M_{4j_r}\Big) = \lambda(M_{4j_1}) \cdots \lambda(M_{4j_r}).$$

Then, there exists a unique multiplicative sequence, $\{K_l\}$, so that for every M of dimension n,

$$\lambda(M) = K_{\frac{n}{4}}(p_1, \ldots, p_{\frac{n}{4}})[M].$$

We get the fundamental theorem:

Theorem 3.54 *(Hirzebruch Signature Theorem) For all real differentiable oriented manifolds, M, we have:*

(1) If $\dim_{\mathbb{R}} M \not\equiv 0 \pmod 4$, then $I(M) = 0$.

(2) If $\dim_{\mathbb{R}} M = 4k$, then

$$I(M) = L_k(p_1, \ldots, p_k)[M].$$

Proof. Recall, I is a function from manifolds to $\mathbb{Z}$ and clearly satisfies (1). By Thom's second Theorem (Theorem 3.52), I satisfies (2). Take as basis sequence: $M_{4k} = \mathbb{P}^{2k}_{\mathbb{C}}$. We have

$$I(M_{4k}) = \sum_{p=0}^{2k} (-1)^p h^{p,q}(M_{4k}),$$

by the Hodge Index Theorem (Theorem 2.77). As $h^{p,p} = 1$ and $h^{p,q} = 0$ if $p \neq q$, we get

$$I(M_{4k}) = 1.$$

Now we further know the Künneth formula for the $h^{p,q}$ of a product (of two, hence any finite number of complex manifolds). Apply this and get (DX)

$$I\Big(\mathbb{P}^{j_1}_{\mathbb{C}} \prod \cdots \prod \mathbb{P}^{j_r}_{\mathbb{C}}\Big) = 1.$$

Therefore, (3) holds. Then, our previous theorem implies $I(M) = K(M)$ for some K, a multiplicative sequence. But, $K(\mathbb{P}^{2k}_{\mathbb{C}}) = 1$, there and we know there is one and only one multiplicative sequence $\equiv 1$ on all $\mathbb{P}^{2k}_{\mathbb{C}}$, it is L. Therefore, $I(M) = L$, as claimed. $\square$

3.5 The Hirzebruch–Riemann–Roch Theorem (HRR)

We can now state and understand the theorem:

Theorem 3.55 *(Hirzebruch–Riemann–Roch) Suppose X is a complex, smooth, projective algebraic variety of complex dimension n. If E is a rank q complex vector bundle on X, then*

$$\chi(X, \mathcal{O}_X(E)) = \kappa_n\big(\mathrm{ch}(E)(t)\mathrm{td}(X)(t)\big)[X].$$

Here,

$$\chi(X, \mathcal{O}_X(E)) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, \mathcal{O}_X(E)).$$

We need to explicate the theorem.

(a) Write it using the Chern roots

$$1 + c_1(E)t + \cdots + c_q(E)t^q = \prod_{i=1}^{q}(1+\gamma_i t), \quad 1 + c_1(X)t + \cdots + c_q(X)t^n = \prod_{j=1}^{n}(1+\delta_j t),$$

and the theorem says

$$\chi(X, \mathcal{O}_X(E)) = \kappa_n\left(\sum_{i=1}^{q} e^{\gamma_i t}\prod_{j=1}^{n}\frac{\delta_j t}{1-e^{-\delta_j t}}\right)[X].$$

(b) Better explication: Use

$$\begin{aligned}\mathrm{td}(X)(t) = 1 &+ \frac{1}{2}c_1(X)t + \frac{1}{12}(c_1^2(X) + c_2(X))t^2 + \frac{1}{24}c_1(X)c_2(X)t^3 \\ &+ \frac{1}{720}(-c_4(X) + c_3(X)c_1(X) + 3c_2^2(X) + 4c_2(X)c_1^2(X) - c_1^4(X))t^4 + O(t^5)\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}(E)(t) = \mathrm{rk}(E) &+ c_1(E)t + \frac{1}{2}(c_1^2(E) - 2c_2(E))t^2 + \frac{1}{6}(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E))t^3 \\ &+ \frac{1}{24}(c_1^4(E) - 4c_1^2(E)c_2(E) + 4c_1(E)c_3(E) + 2c_2^2(E) - 4c_4(E))t^4 + O(t^5).\end{aligned}$$

(A) Case $n = 1$, X = Riemann surface = complex curve; E = rank q vector bundle on X. HRR says:

$$\chi(X, \mathcal{O}_X(E)) = \left(\frac{1}{2}qc_1(X) + c_1(E)\right)[X].$$

Now, $c_1(X) = \chi(X) =$ Euler-Poincaré(X) = (highest Chern class) $= 2 - 2g$ (where g is the genus of X). Also, $c_1(E) = \deg(E)$ $(= \deg \bigwedge^q E)$, so

$$\chi(X, \mathcal{O}_X(E)) = (1-g)\mathrm{rk}(E) + \deg E.$$

Now,

$$\chi(X, \mathcal{O}_X(E)) = \dim H^0(X, \mathcal{O}_X(E)) - \dim H^1(X, \mathcal{O}_X(E));$$

by Serre duality,

$$\dim H^1(X, \mathcal{O}_X(E)) = \dim H^0(X, \mathcal{O}_X(E^D \otimes \omega_X)),$$

so we get

$$\dim H^0(X, \mathcal{O}_X(E)) - \dim H^0(X, \mathcal{O}_X(E^D \otimes \omega_X)) = \deg E + (\mathrm{rk}(E))(1-g).$$

(Note: We proved this before using the Atiyah-Serre Theorem, see Theorem 3.13.)

(i) $E = \mathcal{O}_X =$ trivial bundle, then $\deg E = 0$ and $\mathrm{rk}\, E = 1$. We get

$$\dim H^0(X, \mathcal{O}_X) - \dim H^0(X, \Omega^1_X) = 1 - g.$$

Now, X connected implies $\dim H^0(X, \mathcal{O}_X) = h^{0,1} = 1$, so

$$g = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega^1_X) = h^{1,0}.$$

(ii) $E = \omega_X = \Omega^1_X$, $\mathrm{rk}\, E = 1$ and HRR says

$$\dim H^0(X, \Omega^1_X) - \dim H^0(X, \mathcal{O}_X) = \deg \Omega^1_X + 1 - g.$$

The left hand side is g and $\dim H^0(X, \mathcal{O}_X) = 1$, so

$$\deg \Omega^1_X = 2g - 2.$$

(iii) $E = T_X = \Omega^{1,D}_X$. Then, $\mathrm{rk}\, E = 1$, $\deg E = 2 - 2g$ and HRR says

$$\dim H^0(X, T_X) - \dim H^1(X, T_X) = 2 - 2g + 1 - g.$$

Assume $g \geq 2$, then $\deg T_X = 2 - 2g < 0$. Therefore, $H^0(X, T_X) = (0)$ and so,

$$-\dim H^1(X, T_X) = 3 - 3g,$$

so that

$$\dim H^1(X, T_X) = 3g - 3.$$

Remark: The group $H^1(X, T_X)$ is the space of infinitesimal analytic deformations of X. Therefore, $3g - 3$ is the dimension of the complex space of infinitesimal deformations of X as complex manifold. suppose we know that there was a "classifying" variety of the genus g Riemann surfaces, say $\mathfrak{M}_g$. Then, if X (our Riemann surface of genus g) corresponds to a smooth point of $\mathfrak{M}_g$, then

$$T_{\mathfrak{M}_g, X} = H^1(X, T_X).$$

Therefore, $\dim_{\mathbb{C}} \mathfrak{M}_g = 3g - 3$ (Riemann's computation).

(B) The case $n = 2$, an algebraic surface. Here, HRR says

$$\chi(X, \mathcal{O}_X(E)) = \left(\frac{1}{12}(c_1^2(X) + c_2(X))\mathrm{rk}(E) + \frac{1}{2}c_1(X)c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E))\right)[X].$$

The left hand side is

$$\dim H^0(X, \mathcal{O}_X(E)) - \dim H^1(X, \mathcal{O}_X(E)) + \dim H^0(X, \mathcal{O}_X(E^D \otimes \omega_X)).$$

Take $E =$ trivial bundle, $\mathrm{rk}\, E = 1$, $c_1(E) = c_2(E) = 0$, and we get

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X))[X] = \frac{1}{12}(\mathcal{K}_X^2 + \chi(X))[X],$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X. We proved that this holds iff $I(X) = \frac{1}{3}p_1(X) = L_1(p_1)[X]$ (see Section 2.6, just after Theorem 2.82). By the Hirzebruch signature theorem, our formula is OK.

Observe, if we take ω_X, not $\mathcal{O}_X$, then the left hand side, $\chi(X, \mathcal{O}_X)$, is

$$\dim H^0(X,\omega_X) - \dim H^1(X,\omega_X) + \dim H^2(X,\omega_X) = \dim H^2(X,\mathcal{O}_X) - \dim H^1(X,\mathcal{O}_X) + \dim H^0(X,\omega_X)$$

(by Serre duality) and the left hand side stays the same.

Take $E = T_X$; $\mathrm{rk}\, E = 2$, $c_1(E) = c_1(X)$, $c_2(E) = c_2(X)$ and the right hand side of HRR is

$$\begin{aligned}\left(\frac{2}{12}(c_1^2(X) + c_2(X)) + \frac{1}{2}c_1^2(X) + \frac{1}{2}c_1^2(E) - c_2(X)\right)[X] &= \left(\frac{7}{6}c_1^2(X) - \frac{5}{6}c_2(X)\right)[X] \\ &= \left(\frac{7}{6}\mathcal{K}_X^2 - \frac{5}{6}\chi(X)\right)[X].\end{aligned}$$

The left hand side of HRR is

$$\dim H^0(X, T_X) - \dim H^1(X, T_X) + \dim H^0(X, T_X^D \otimes \omega_X).$$

Now,

$$T_X^D \otimes T_X^D \longrightarrow T_X^D \wedge T_X^D = \omega_X$$

gives by duality

$$\begin{aligned} T_X^D &\cong \mathrm{Hom}(T_X^D, \omega_X) \\ &\cong \mathrm{Hom}(T_X^D \otimes \omega_X^D, \mathcal{O}_X) \\ &\cong T_X \otimes \omega_X, \end{aligned}$$

so the left hand side is

dim(global holo vector fields on X) $-$ dim(infinitesimal deformations of X) $+$ dim(global section of $T_X \otimes \omega_X^{\otimes 2}$).

Take $E = \Omega_X^1 = T_X^D$, $\mathrm{rk}\, E = 2$, $c_1(E) = c_1(\omega_X) = -c_1(T_X) = -c_1(X)$, $c_2(E) = c_2(X)$. The right hand side of HRR is

$$\frac{2}{12}(c_1^2(X) + c_2(X)) - \frac{1}{2}c_1^2(X) + \frac{1}{2}c_1^2(X) - c_2(X) = \frac{1}{6}c_1^2(X) - \frac{5}{6}c_2(X).$$

The left hand side of HRR is

$$\dim H^0(X,\Omega_X^1) - \dim H^1(X,\Omega_X^1) + \dim H^2(X,\Omega_X^1) = h^{1,0} - h^{1,1} + h^{1,2} = h^{1,0} - h^{1,1} + h^{1,0} = b_1(X) - h^{1,1}.$$

It follows that

$$b_1(X) - h^{1,1} = \left(\frac{7}{6}\mathcal{K}_X^2 - \frac{5}{6}\chi(X)\right)[X],$$

so

$$b_1(X) - \left(\frac{7}{6}\mathcal{K}_X^2 - \frac{5}{6}\chi(X)\right)[X] = h^{1,1}.$$

Also,

$$\begin{aligned} H^0(X,\Omega_X^1) &= H^2(X, \omega_X \otimes T_X)^D \\ H^1(X,\Omega_X^1) &= H^1(X, \omega_X \otimes T_X)^D \\ H^2(X,\Omega_X^1) &= H^0(X, \omega_X \otimes T_X)^D \end{aligned}$$

and we get no new information.

When we know something about X, we can say more. For example, say X is a hypersurface of degree d in $\mathbb{P}^3_{\mathbb{C}}$. Then, write

$$H \cdot X = h = i^* H,$$

where $i\colon X \to \mathbb{P}^3_{\mathbb{C}}$. We know

$$\mathcal{N}_{X \hookrightarrow \mathbb{P}^3} = \mathcal{O}_X(d \cdot h),$$

so

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^3} \restriction X \longrightarrow \mathcal{O}_X(dh) \longrightarrow 0 \quad \text{is exact.}$$

We have

$$(1 + c_1(X)t + c_2(X)t^2)(1 + dht) = (1 + Ht)^4 \restriction X = (1 + ht)^4,$$

so

$$1 + c_1(X)t + c_2(X)t^2 = (1 + 4ht + 6h^2t^2)(1 - dht + d^2h^2t^2) = 1 + (4 - d)ht + (6 - 4d + d^2)h^2t^2.$$

So $c_1(X) = (4 - d)h$ and $c_2(X) = (6 - 4d + d^2)h^2$. Now,

$$h^2[X] = i^*(H \cdot X) i^*(H \cdot X) = H \cdot H \cdot X = \deg X = d.$$

Consequently,

$$\chi(X, \mathcal{O}_X(E)) = \frac{1}{12}\mathrm{rk}(E)((4 - d)^2 d + (6 - 4d + d^2)d) + \frac{1}{2}c_1(E)(4 - d)h[X] + \frac{1}{2}(c_1^2(E) - 2c_2(E))[X].$$

Take eH and set $E =$ line bundle $eh = eH \cdot X = eH \restriction X = \mathcal{O}_X(e)$. In this case, $\mathrm{rk}(E) = 1$, $c_2(E) = 0$ and $c_1(E) = eh$. We get

$$\chi(X, \mathcal{O}_X(e)) = \frac{1}{6}(11 - 6d + d^2)d + \frac{1}{2}e(4 - d)d + \frac{1}{2}e^2 d,$$

i.e.,

$$\chi(X, \mathcal{O}_X(e)) = \left(\frac{1}{6}(11 - 6d + d^2) + \frac{1}{2}(e^2 - ed + 4e)\right) d.$$

(C) $X =$ abelian variety $=$ projective group variety.

As X is a group, T_X is the trivial bundle, so $c_1(X) = c_2(X) = 0$. When X is an abelian surface we get

$$\chi(X, \mathcal{O}_X(E)) = \frac{1}{2}(c_1^2(E) - 2c_2(E))[X].$$

When X is an abelian curve $=$ elliptic curve ($g = 1$), we get

$$\chi(X, \mathcal{O}_X(E)) = c_1(E) = \deg E.$$

Say the abelian surface is a hypersurface in $\mathbb{P}^3_{\mathbb{C}}$. We know $c_1(X) = 0$ and $c_2(X) = (4 - d)h$. This implies $d = 4$, but $c_2(X) = 6h^2 \neq 0$, a contradiction! Therefore, no abelian surface in $\mathbb{P}^3_{\mathbb{C}}$ is a hypersurface.

Now, assume $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$, where $N > 3$ and X is an abelian surface. Set $E = \mathcal{O}_X(h)$ and compute $\chi(X, \mathcal{O}_X(h))$, where $h = H \cdot X$. We have $c_1(\mathcal{O}_X(h)) = h$ and $c_2(\mathcal{O}_X(h)) = 0$. Then,

$$c_1^2(E)[X] = h^2[X] = H \cdot H \cdot X = \deg X$$

as subvariety of $\mathbb{P}^N_{\mathbb{C}}$. HRR for abelian surfaces embedded in $\mathbb{P}^N_{\mathbb{C}}$ with $N > 3$ yields

$$\chi(X, \mathcal{O}_X(1)) = \frac{1}{2}\deg X.$$

As the left hand side is an integer, we deduce that $\deg X$ must be even.

(D) $X = \mathbb{P}^n_{\mathbb{C}}$. From

$$1 + c_1(X)t + \cdots + c_n(X)t^n = (1 + Ht)^{n+1}$$

we deduce

$$\delta_1 = \cdots = \delta_{n+1} = H.$$

Take

$$1 + c_1(X)t + \cdots + c_n(X)t^n = \prod_j (1 + \gamma_j t)$$

and look at $E \otimes H^{\otimes r} = E(r)$. We have

$$\begin{aligned}
\chi(\mathbb{P}^n, \mathcal{O}_X(E(r))) &= \kappa_n \left(\sum_{i=1}^q e^{(\gamma_i + r)t} \frac{(Ht)^n}{(1 - e^{-Ht})^n} \right) [X] \\
&= \sum_{l=1}^q \frac{1}{2\pi i} \int_C \frac{e^{(\gamma_l + r)Ht}}{(1 - e^{-Ht})^{n+1}} \, d(Ht) \\
&= \sum_{l=1}^q \frac{1}{2\pi i} \int_C \frac{e^{(\gamma_l + r)z}}{(1 - e^{-z})^{n+1}} \, d(z),
\end{aligned}$$

where C is a small circle. Let $u = 1 - e^{-z}$, then $du = e^{-z}dz = (1 - u)dz$, so

$$dz = \frac{du}{1 - u}.$$

We also have $e^{(\gamma_l + r)z} = (e^{-z})^{-(\gamma_l + r)} = (1 - u)^{-(\gamma_l + r)}$. Consequently, the integral is

$$\sum_{l=1}^q \frac{1}{2\pi i} \int_{u=0}^{2\pi} \frac{du}{(1 - u)^{\gamma_l + r + 1} u^{n+1}}$$

where is the path of integration is a segment of the line $z = \epsilon + iu$. It turns out that

$$\frac{1}{2\pi i} \int_{u=0}^{2\pi} \frac{du}{(1 - u)^{\gamma_l + r + 1} u^{n+1}} = \beta(\gamma_l, n) = \binom{n + \gamma_l + r}{n}$$

so HRR implies

$$\chi(\mathbb{P}^n, \mathcal{O}_X(E(r))) = \sum_{l=1}^q \binom{n + \gamma_l + r}{n} \in \mathbb{Q}.$$

But, the right hand side has denominator $n!$ and the left hand side is an integer. We deduce that for all $r \in \mathbb{Z}$, for all $n \geq$ and all $q \geq 1$,

$$\sum_{l=1}^q \binom{n + \gamma_l + r}{n} \in \mathbb{Z}.$$

(Here, $1 + c_1(E)Ht + \cdots + c_q(E)(Ht)^q = \prod_{j=1}^q (1 + \gamma_j Ht)$.)

Take $r = 0, q = 2$. We get

$$\binom{n + \gamma_1}{n} + \binom{n + \gamma_2}{n} \in \mathbb{Z}.$$

For $n = 2$, we must have

$$(2 + \gamma_1)(1 + \gamma_1) + (2 + \gamma_2)(1 + \gamma_2) \equiv 0 \ (2),$$

i.e.,

$$2 + 3\gamma_1 + \gamma_1^2 + 2 + 3\gamma_2 + \gamma_2^2 \equiv 0 \ (2),$$

which is equivalent to

$$3c_1 + c_1^2 - 2c_2 \equiv 0 \ (2).$$

Thus, we need $c_1(3 + c_1) \equiv 0 \ (2)$, which always holds.

Now, take $n = 3$. We have

$$\binom{3+\gamma_1}{3} + \binom{3+\gamma_2}{3} \in \mathbb{Z},$$

i.e.,

$$(3 + \gamma_1)(2 + \gamma_1)(1 + \gamma_1) + (3 + \gamma_2)(2 + \gamma_2)(1 + \gamma_2) \equiv 0 \ (6).$$

This amounts to

$$(6 + 5\gamma_1 + \gamma_1^2)(1 + \gamma_1) + (6 + 5\gamma_2 + \gamma_2^2)(1 + \gamma_2) \equiv 0 \ (6)$$

which is equivalent to

$$\gamma_1(5 + \gamma_1)(1 + \gamma_1) + \gamma_2(5 + \gamma_2)(1 + \gamma_2) \equiv 0 \ (6),$$

i.e.,

$$\gamma_1(5 + 6\gamma_1 + \gamma_1^2) + \gamma_2(5 + 6\gamma_2 + \gamma_2^2) \equiv 0 \ (6)$$

which can be written in terms of the Chern classes as

$$5c_1 + 6(c_1^2 - 2c_2) + c_1^3 - 3c_1c_2 \equiv 0 \ (6),$$

i.e.,

$$c_1(c_1^2 - 3c_2 + 5) \equiv 0 \ (6).$$

Observe that

$$c_1^3 + 5c_1 \equiv 0 \ (6)$$

always, so we conclude that c_1c_2 must be *even*.

Say $i\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ is an embedding of $\mathbb{P}^2_{\mathbb{C}}$ into $\mathbb{P}^3_{\mathbb{C}}$.

Question: Does there exist a rank 2 bundle on $\mathbb{P}^3_{\mathbb{C}}$, say E, so that $i^*(E) = T_{\mathbb{P}^2_{\mathbb{C}}}$.?

If so, E has Chern classes c_1 and c_2 and

$$c_1(T_{\mathbb{P}^2_{\mathbb{C}}}) = i^*(c_1), \quad c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = i^*(c_2).$$

This implies

$$c_1c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = i^*(c_1c_2(E)),$$

which is even (case $n = 3$). But,

$$c_1(T_{\mathbb{P}^2_{\mathbb{C}}}) = 3H_{\mathbb{P}^2}, \quad c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = 3H_{\mathbb{P}^2},$$

so

$$c_1c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = 9H^2,$$

which is **not** even! Therefore, the answer is no.

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