Choice of Riemannian Metrics for Rigid Body Kinematics

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Abstract

The set of rigid body motions forms a Lie group called $SE(3)$, the special Euclidean group in three dimensions. In this paper we investigate possible choices of Riemannian metrics and affine connections on this manifold. In the first part of the paper we derive the semi-Riemannian metrics whose geodesics are screw motions and show that the only metrics are indefinite but non degenerate, and they are unique up to a choice of two scaling constants. In the second part of the paper we investigate affine connections which through the covariant derivative give the correct expression for the acceleration of a rigid body. We prove that there is a unique symmetric connection which achieves this. Further, we show that there is a family of metrics that are compatible with such connection. A metric in this family must be a product of the bi-invariant metric on the group of rotations and a positive definite constant metric on the group of translations.

1 Introduction

The set of all three-dimensional rigid body displacements forms a Lie group [1, 2]. This group is generally referred to as $SE(3)$, the special Euclidean group in three dimensions. The tangent space at the identity endowed with the Lie bracket operation has the structure of a Lie algebra and is denoted by $se(3)$. It is isomorphic to the set of all twists and the Lie bracket of two twists corresponds to the motor product of the respective motors.

There is extensive literature on the algebra of twists and the theory of screws [3, 4, 5, 6]. It is well known that there is an inner product on the space of twists induced by the usual inner product on $\mathbb{R}^6$ but this inner product is not invariant under change of coordinate frame [5, 7]. On the space of twists, there are two quadratic forms, the Killing form and the Klein form that are bi-invariant [2]: They are invariant under change of the inertial reference frame and change of the body-fixed reference frame. However neither form is positive definite: The Killing form is degenerate and the Klein form is indefinite.

Because of the Lie group structure of $SE(3)$ it is possible to define a positive definite quadratic form or a metric on the tangent space at the identity and extend it to the whole group by left translation. Such a metric is called left invariant and is invariant with respect to change in inertial frame (but not with respect to change in body-fixed frame). Similarly, one can define a right invariant metric through right translation and this metric is invariant with respect to change in body-fixed frame. In this way we can endow $SE(3)$ with a Riemannian metric and give it the structure of a Riemannian manifold. If we only require the quadratic form to be non-degenerate (instead of positive definite), the resulting metric is called a semi-Riemannian metric. Park and Brockett [8] propose a left invariant Riemannian metric, which when restricted to the group of rotations, $SO(3)$, is bi-invariant, and further preserves the isotropy of $\mathbb{R}^3$. However, the metric depends on the choice of a length scale. A good discussion on the geometry of $SE(3)$ can be found in the appendix of [9].

A Riemannian metric imposes quite strong structure on the manifold. We can obtain somewhat less restrictive structure through a connection. In particular, an affine connection defines the notion of parallelism (which is trivial in Euclidean space) on the manifold by defining a rule for parallel transport along curves. The notion of parallelism leads to definition of the covariant derivative of a vector field (the generalization of directional derivatives in Euclidean space) along a curve. In the context of kinematics, the motion of a rigid body is a curve on $SE(3)$ and the velocity at any point is the tangent to the curve at that point. We need the definition of a covariant derivative before we can talk about the acceleration of the rigid body: The acceleration is the covariant derivative of the velocity field along the curve.

Given a Riemannian metric, a theorem due to Levi-Civita guarantees the existence and uniqueness of a special connection which is compatible with the metric and symmetric. This connection is called a Riemannian connection. The connection determines the geodesics, the generalization of straight lines of Euclidean geometry to Riemannian manifolds. Park [10] derives geodesics for the scale-dependent left invariant metric on $SE(3)$ using covering maps. Žefran *et al.* [11] obtain the same result directly from the connection. They also find minimum acceleration and minimum jerk trajectories on $SE(3)$ and develop expressions for the acceleration of a rigid body.

This paper addresses the choice of metrics and connections for $SE(3)$. Some of the geodesics from the scale dependent left invariant metric, are screw motions [11]. Since Chasles theorem guarantees the existence of a screw motion (and the uniqueness of the screw axis) between any two points on $SE(3)$, a natural question is whether there exists a metric for which all the geodesics are screw motions. We show that there is a semi-Riemannian metric which is uniquely determined up to two scale constants, for which the screw motions are geodesics. The second question that we ask is whether there is a unique connection that gives the physically correct acceleration for a rigid body motion. We show that there is a unique symmetric connection which is physically meaningful (from the point of view of acceleration analysis). Further, this connection can be derived from a family of Riemannian metrics. Any element of this family of metrics is a product of a bi-invariant metric on the group of rotations and a positive definite constant metric on the group of translations.

The paper is organized as follows. At the beginning we briefly discuss notions of a Riemannian manifold, a metric and a connection that are later used in the derivations. In the next section, we look for metrics for which screw motions are geodesics. We arrive at a system of partial differential equations for the metric coefficients. The solution of the system follows from the conditions on the consistency of the system. Next, we study affine connections which lead to the acceleration used in rigid-body kinematics. By requiring symmetry of the connection, we are able to show that such connection is unique. Further, we find a family of metrics which are consistent with the connection. We arrive at the resulting family of metrics through the study of geodesics. We conclude the paper with a short discussion.

2 Kinematics, Lie groups and differential geometry

2.1 The Lie group $SE(3)$

Consider a rigid body moving in free space. Assume any inertial reference frame F fixed in space and a frame M fixed to the body at point O' as shown in Figure 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame F to frame M. The set of all such matrices is called $SE(3)$, the special Euclidean group of rigid body transformations in three-dimensions. It is not difficult to show that $SE(3)$ is a group for the standard matrix multiplication and that it can be endowed with a differentiable structure. It is therefore a Lie group [12].

Figure 1: The inertial (fixed) frame and the moving frame attached to the rigid body

On any Lie group, G, the tangent space at the group identity, T_IG, has the structure of a Lie algebra. A short calculation shows that the Lie algebra of $SE(3)$, denoted by $se(3)$, is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix}, \Omega \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}. \tag{1}$$

A 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for an arbitrary vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where $\times$ is the vector cross product operation in $\mathbb{R}^3$. Each element $T \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$ or a 6×1 vector $\left[\omega^T, v^T\right]^T$. In the paper, both notations will be used interchangeably.

Given a curve $A(t) : [-a, a] \to SE(3)$, an element of the Lie algebra, $T(t)$, can be attributed to the tangent vector $\dot{A}(t)$ at an arbitrary point t by:

$$T(t) = A^{-1}(t)\dot{A}(t). \tag{2}$$

A curve on $SE(3)$ physically represents a motion of the rigid body. If $\{\omega(t), v(t)\}$ is the vector corresponding to $T(t)$, then ω physically corresponds to the angular velocity of the rigid body while v is the linear velocity of the origin O' of the frame M fixed to the rigid body, both expressed in the frame M. In kinematics, elements of this form are called twists [13]. Thus, $se(3)$ is isomorphic to the set of twists. For this reason, elements of $se(3)$ will be called twists. It is easy to check that the twist $T(t)$ computed from Eq. (2) does not depend on the choice of the inertial frame F. For this reason, $T(t)$ is called the left invariant representation of the tangent vector $\dot{A}$. Alternatively, a tangent vector $\dot{A}$ can be identified with a right invariant twist (invariant with respect to the choice of the body-fixed frame M). In this paper we concentrate on the left invariant twists but the derivations for the right invariant twists are analogous.

Since $se(3)$ is a vector space any element can be expressed as a 6×1 vector of components corresponding to a chosen basis. The standard basis that will be used throughout the paper is:

$$
\begin{array}{lll}
L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
L_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\tag{3}
$$

Twists L_1, L_2 and L_3 represent instantaneous rotations about while L_4, L_5 and L_6 correspond to the instantaneous translations along the Cartesian axes x, y and z, respectively. The chosen basis has the useful property that the components of a twist $T \in se(3)$ are given precisely by the pair of the velocities, $\{\omega, v\}$.

The Lie bracket of two elements $T_1, T_2 \in se(3)$ is defined by:

$$[T_1, T_2] = T_1 T_2 - T_2 T_1.$$

It can be easily verified that if $\{\omega_1, v_1\}$ and $\{\omega_2, v_2\}$ are vectors of components corresponding to twists T_1 and T_2, the vector pair $\{\omega, v\}$ corresponding to their Lie bracket $[T_1, T_2]$ is given by

$$\begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}. \tag{4}$$

In kinematics, this expression is known as the motor product of the two twists.

The Lie bracket of two elements of a Lie algebra is an element of the Lie algebra. Thus, it can be expressed as a linear combination of the basis vectors. The coefficients C_{ij}^k corresponding to the Lie brackets of the basis vectors, defined by

$$[L_i, L_j] = \sum_k C_{ij}^k L_k. \tag{5}$$

are called structure constants of the Lie algebra [12]. For $se(3)$, they can be directly computed from Equation (4). The nonzero structure constants for the basis (3) are:

$$
\begin{array}{l}
C_{12}^3 = C_{31}^2 = C_{23}^1 = C_{15}^6 = C_{26}^4 = C_{34}^5 = C_{42}^6 = C_{53}^4 = C_{61}^5 = 1 \\
C_{21}^3 = C_{13}^2 = C_{32}^1 = C_{51}^6 = C_{62}^4 = C_{43}^5 = C_{24}^6 = C_{35}^4 = C_{16}^5 = -1
\end{array}
\tag{6}
$$

2.2 Left invariant vector fields

A differentiable vector field on a manifold is a smooth assignment of a tangent vector to each element of the manifold. On $SE(3)$, an example of a differentiable vector field, X, is obtained by left translation of an element of $se(3)$:

$$X(A) = \hat{T}(A) = AT, \tag{7}$$

where $T \in se(3)$. Such a vector field is called a left invariant vector field. We use the notation $\hat{T}$ to indicate that the vector field is obtained by left translating the Lie algebra element T. The set of all left invariant vector fields is a vector space and by construction it is isomorphic to $se(3)$. A similar approach can be used to derive right invariant vector fields. In general, a vector field need not be left or right invariant.

Since vectors $L_1, L_2, \ldots, L_6$ are a basis for the Lie algebra $se(3)$, at any point $A \in SE(3)$ the vectors $\hat{L}_1(A), \ldots, \hat{L}_6(A)$ obtained from Eq. (7) form a basis of the tangent space at that point. Therefore, any vector field X can be expressed as

$$X = \sum_{i=1}^{6} X^i \hat{L}_i, \tag{8}$$

where the coefficients X^i vary over the manifold. If the coefficients are constants, then X is left invariant. Equation (8) suggests that we associate a vector pair $\{\omega, v\}$ defined by

$$\omega = [X^1, X^2, X^3]^T, \qquad v = [X^4, X^5, X^6]^T.$$

to an arbitrary vector field X. We also have [14]:

$$[\hat{L}_i, \hat{L}_j] = \widehat{[L_i, L_j]} = \sum_k C_{ij}^k \hat{L}_k. \tag{9}$$

2.3 Exponential mapping and local coordinates

Since $SE(3)$ is a 6-dimensional manifold, it can be locally described as a diffeomorphic image of an open set in $\mathbb{R}^6$ [14]. A particularly convenient representation of $SE(3)$ can be derived from the exponential map $\exp : se(3) \rightarrow SE(3)$. For $T \in se(3)$, the exponential map is given by the usual matrix exponentiation:

$$\exp(T) = \sum_{k=0}^{\infty} \frac{T^k}{k!}. \tag{10}$$

The explicit formula for the exponential map restricted to $SO(3)$ is given by the Rodrigues' formula [9].

The function exp is a surjective map of $se(3)$ onto $SE(3)$ [9]. However, for a small enough neighborhood of zero in $se(3)$ it is a diffeomorphism onto a neighborhood of the identity element in $SE(3)$. We can thus define local coordinates, ξ^i, for $A \in SE(3)$ sufficiently close to the identity:

$$A(\xi) = \exp\left(\xi^1 L_1 + \xi^2 L_2 + \xi^3 L_3 + \xi^4 L_4 + \xi^5 L_5 + \xi^6 L_6\right). \tag{11}$$

The coordinates ξ^i are called canonical coordinates of the first kind [15].

2.4 Riemannian metrics on Lie groups

If a smoothly varying, positive definite, bilinear, symmetric form is defined on the tangent space at each point on the manifold, we say the manifold is Riemannian. The bilinear form is an inner product on the tangent space and is called a Riemannian metric.

On a Lie group, an inner product in the tangent space at the identity can be extended to a Riemannian metric (everywhere on the manifold) using the idea of left translation. Assume that the inner product of two elements $T_1, T_2 \in se(3)$ is defined by

$$< T_1, T_2 >_I = t_1^T W t_2, \tag{12}$$

where t_1 and t_2 are the 6×1 vectors of components of T_1 and T_2 with respect to some basis and W is a positive definite matrix. If V_1 and V_2 are tangent vectors at an arbitrary group element $A \in SE(3)$, the inner product $< V_1, V_2 >_A$ in the tangent space $T_A SE(3)$ can be defined by:

$$< V_1, V_2 >_A = < A^{-1} V_1, A^{-1} V_2 >_I . \tag{13}$$

The metric obtained in such a way is called a left invariant metric [14]. A right invariant Riemannian metric can be defined in a similar way.

2.5 Affine connection and covariant derivative

The motion of a rigid body can be represented by a curve, $A(t)$, on $SE(3)$. The velocity at an arbitrary point is the tangent vector to the curve at that point. In order to obtain other kinematic quantities, such as acceleration and jerk, or to engage in a dynamic analysis, we need to be able to differentiate a vector field along the curve. If the manifold $SE(3)$ is embedded in the space of all 4×4 matrices differentiation is a straightforward process. However, we would like to obtain definitions for the derivative that are intrinsic, that is, they depend only on the manifold $SE(3)$ itself and not on the ambient space. Further, we would like to obtain a definition that is invariant with respect to the choice of local coordinate systems.

Differentiable structure of the manifold does not provide any means to compare values of a vector field at different points. At each point, $A \in SE(3)$, the value of a vector field belongs to the tangent space $T_A\, SE(3)$ and tangent spaces at different points are not related. To be able to differentiate a vector field, we have to define how to compare vectors that belong to different tangent spaces. More precisely, to differentiate a vector field along a curve, we must be able to compare vectors from tangent spaces at different points on the curve. For this purpose, it suffices to define when two vectors that belong to tangent spaces at different points of a curve are parallel. The affine connection formalizes the notion of parallelism: Given a point A on a curve γ and a vector $V \in T_A\, SE(3)$, the affine connection assigns to each other point $A' \in \gamma$ a vector $V' \in T_{A'}\, SE(3)$. By definition, V' is parallel to V along the curve γ and is called *parallel transport of* V *along* γ.

With parallel transport, we can compare two vectors at different points of the curve simply by transporting them to the same point on a curve. In particular, we can define a derivative of a vector field along a curve $\gamma(t)$. Let X be a vector field defined along γ, $A = \gamma(t_0)$ and $V = X(A)$. Denote by $X^{t_0}(t)$ the parallel transport of the vector $X(\gamma(t))$ to the point $A = \gamma(t_0)$. The *covariant derivative* of X along γ is:

$$\frac{DX}{dt}(A) = \lim_{t \to t_0} \frac{X^{t_0}(t) - X(A)}{t}. \tag{14}$$

If $Y = \frac{d\gamma}{dt}$ is the tangent vector field for the curve γ, the covariant derivative is also denoted by:

$$\frac{DX}{dt} = \nabla_Y X. \tag{15}$$

It is important to note that in order to compute the covariant derivative of a vector field along a curve it suffices to know the values of the vector field on the curve, it does not have to be defined elsewhere. If a covariant derivative of a vector field X along a curve $\gamma(t)$ vanishes, then X is equal to its parallel transport along γ.

Covariant derivative of a vector field is another vector field so it can be expressed as a linear combination of the basis vector fields. The coefficients, Γ^k_{ji}, of the covariant derivative of a basis vector field along another basis vector field,

$$\nabla_{\hat{L}_i} \hat{L}_j = \sum_k \Gamma^k_{ji} \hat{L}_k,$$

are called Christoffel symbols[1]. Note the reversed order of indices i and j.

The velocity, $V(t)$, of the rigid body describing the motion $A(t)$ is given by the tangent vector field along the curve:

$$V(t) = \frac{dA(t)}{dt}$$

The acceleration of the rigid body is the covariant derivative of the velocity along the curve

$$\frac{\mathrm{D}}{dt}\left(\frac{dA}{dt}\right) = \nabla_V V$$

Note that the acceleration depends on the choice of the connection.

Given a Riemannian manifold, there exists a unique connection, called *Levi-Civita* or *Riemannian connection*, which is compatible with the metric and symmetric [14]. That is, given vector fields X, Y and Z,

(a) $X <Y, Z> = <\nabla_X Y, Z> + <Y, \nabla_X Z>$ (*compatibility with the metric*),

(b) $\nabla_X Y - \nabla_Y X = [X, Y]$ (*symmetry*).

[1] In the literature, different definitions for the Christoffel symbols can be found. Some texts (e.g. [14]) reserve the term for the case of the coordinate basis vectors. We follow the more general definition from [16] in which the basis vectors can be arbitrary.

3 Metrics and screw motions

3.1 Screw motions and displacements on $SE(3)$

One of the fundamental results in rigid body kinematics [13] states that any displacement of a rigid body can be realized in a special canonical way and was proved by Chasles at the beginning of the 19th century:

Theorem 3.1 (Chasles) *Any rigid body displacement can be realized by a rotation about an axis combined with a translation parallel to that axis. This displacement is called a* screw displacement *and the axis along which the displacement occurs is known as a* screw axis.

A rigid body motion in which the rigid body rotates with a constant rotational velocity about an axis while concurrently translating with a constant translational velocity along that axis is called *a screw motion.* Strictly speaking, a displacement is different from a motion: A motion is a curve on $SE(3)$ while a displacement is an element of $SE(3)$.

Another family of curves of particular interest on $SE(3)$ are *one-parameter subgroups.* On a Lie group, a curve $A(t)$ is a one-parameter subgroup, if it is a subgroup and if the following identity holds:

$$A(t_1 + t_2) = A(t_1) \circ A(t_2), \tag{16}$$

where $\circ$ denotes the group operation on $SE(3)$. One-parameter subgroups on $SE(3)$ are given by [14]:

$$A(t) = \exp(t\, S) \tag{17}$$

where S is an element of $se(3)$. Therefore, a one-parameter subgroup can be viewed as a curve whose tangent vector field is a left invariant vector field obtained from the Lie algebra element S. It is not difficult to see that physically, a one-parameter subgroup represents a screw motion. Therefore, in the language of differential geometry, the Chasles theorem becomes:

Theorem 3.2 (Chasles restated) *For every element in $SE(3)$ there is a one-parameter subgroup to which that element belongs. Except for the identity, the screw axis for every element is unique, but there are multitude of screw motions along that axis which contain the element.*

The following corollary immediately follows:

Corollary 3.3 *Given any two distinct elements, A_1 and A_2, the following is true:*

1. *There is a unique one-parameter subgroup, $\gamma_L(t) = \exp(t\, S_L)$, which when left translated by A_1 contains A_2:*

$$A_L(t) = A_1 \;\exp(t\, S_L), \qquad\qquad A_2 = A_L(1) = A_1 \exp(S_L).$$

2. *There is a unique one-parameter subgroup, $\gamma_R(t) = \exp(t\, s_R)$, which when right translated by A_1 contains A_2:*

$$A_R(t) = \exp(t\, S_R) A_1, \qquad\qquad A_2 = A_R(1) = \exp(S_R) A_1.$$

S_L and S_R both belong to $se(3)$. Since

$$A_1 \exp(t\, S_L) = A_1 \sum_{k=0}^{\infty} \frac{t^k\, S_L^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k (A_1\; S_L A_1^{-1})^k}{k!}\; A_1, \tag{18}$$

it follows that:

$$S_R = A_1 S_L A_1^{-1} = Ad_{A_1} S_L,$$

and

$$A_R(t) = A_L(t).$$

3.2 Screw motions as geodesics

Given a Riemannian manifold (a manifold with a Riemannian metric), Σ, the length of a smooth curve $A : [a, b] \to \Sigma$, is defined as

$$L(A) = \int_a^b < \frac{dA}{dt}, \frac{dA}{dt} >^{\frac{1}{2}} dt \tag{19}$$

Among all the curves connecting two points, we are usually interested in the curve of minimal length. If a minimum length curve between two points exists, the curve must be a critical point of L. Critical points of the functional L are given by [14]:

$$\nabla_{\frac{dA}{dt}} \frac{dA}{dt} = 0. \tag{20}$$

Curves that satisfy this equation are called *geodesics* and the equation itself is called the *geodesic equation.* From what we said in 2.5, we see that geodesics are curves for which velocity is a parallel vector field. There are cases when a geodesic between two arbitrary points does not exist. Furthermore, there could be more than one geodesic connecting two points.

Given that any two elements of $SE(3)$ can be connected with a screw motion, it is natural to ask whether screw motions can be geodesics. More precisely, is there a Riemannian metric for which screw motions are geodesics.

We have seen that for a screw motion $\gamma(t) = \exp(t\,S)$ the tangent vector field $\frac{d\gamma}{dt}$ is a left invariant vector field corresponding to $S \in se(3)$. This means that components of S with respect to the basis $\hat{L}_i$ are constant. If this trajectory is to be a solution to Eq. (20), we have

$$\nabla_{S^i \hat{L}_i} S^j \hat{L}_j = \frac{dS^i}{dt} \hat{L}_j + S^i S^j \nabla_{\hat{L}_i} \hat{L}_j = S^i S^j \nabla_{\hat{L}_i} \hat{L}_j = 0.$$

The above equation is satisfied if and only if

$$\nabla_{\hat{L}_i} \hat{L}_j + \nabla_{\hat{L}_j} \hat{L}_i = 0.$$

Since ∇ is a metrical connection, it is symmetric:

$$\nabla_{\hat{L}_i} \hat{L}_j - \nabla_{\hat{L}_j} \hat{L}_i = [\hat{L}_i, \hat{L}_j].$$

It immediately follows that:

$$\nabla_{\hat{L}_i} \hat{L}_j = \frac{1}{2} [\hat{L}_i, \hat{L}_j]. \tag{21}$$

Further, ∇ must be compatible with the metric, so we have

$$\hat{L}_k < \hat{L}_i, \hat{L}_j > = < \nabla_{\hat{L}_k} \hat{L}_i, \hat{L}_j > + < \hat{L}_i, \nabla_{\hat{L}_k} \hat{L}_j > . \tag{22}$$

Let $g_{ij} = < \hat{L}_i, \hat{L}_j >$. The last equation implies

$$\hat{L}_k(g_{ij}) = \frac{1}{2} \Big(< [\hat{L}_k, \hat{L}_i], \hat{L}_j > + < \hat{L}_i, [\hat{L}_k, \hat{L}_j] > \Big) . \tag{23}$$

Since $\hat{L}_i$ are left invariant vector fields, $[\hat{L}_i, \hat{L}_j] = [L_i, L_j]$ and $< \hat{L}_i, \hat{L}_j > = < L_i, L_j >$ [14]. Furthermore, the Lie bracket can be expressed using the structure constants:

$$[L_i, L_j] = \sum_k C_{ij}^k L_k. \tag{24}$$

Equation (23) therefore becomes:

$$\hat{L}_k(g_{ij}) = \frac{1}{2} (C_{ki}^l g_{lj} + C_{kj}^l g_{li}). \tag{25}$$

We have arrived at the following:

Proposition 3.4 *Screw motions will satisfy the geodesic equation (20) for a Riemannian metric $G = \{g_{ij}\}$ if and only if the coefficients g_{ij} satisfy Eq. (25).*

Metric coefficients are symmetric. Since $SE(3)$ is a 6 dimensional manifold, there are 21 coefficients defining the metric: $\{g_{ij} | i \leq j, j \leq 6\}$. Furthermore, we have 6 basis vectors. Equation 25 thus expands to a total of 126 equations. These are partial differential equations because each vector field represents a derivation. The system of equations has to be solved for the metric coefficients g_{ij}. The complete set of equations is given in Appendix A.

To find the solution we will use the following Lemma:

Lemma 3.5 *Given a set of partial differential equations*

$$X(f) = g_x \tag{26}$$
$$Y(f) = g_y \tag{27}$$
$$Z(f) = g_z \tag{28}$$

where X, Y, and Z are vector fields such that $Z = [X, Y]$ and f, g_x, g_y and g_z are differentiable (real valued) functions, the solution exists only if

$$X(g_y) - Y(g_x) = g_z. \tag{29}$$

Proof: By applying X on Eq. (27), Y on Eq. (26) and subtracting the two resulting equations, we get:

$$X\,Y(f) - Y\,X(f) = X(g_y) - Y(g_x). \tag{30}$$

But the left-hand side is by definition $[X, Y](f)$, which is by assumption equal to $Z(f)$. Equation (29) then follows from Eq. (28).

■

We now state the first major result of this paper:

Theorem 3.6 *A matrix of coefficients $G = \{g_{ij}\}$ satisfies the system of partial differential equations (25) if and only if it has the form*

$$G = \begin{bmatrix} \alpha\, I_{3\times3} & \beta\, I_{3\times3} \\ \beta\, I_{3\times3} & 0_{3\times3} \end{bmatrix}, \tag{31}$$

where α and β are constants.

Proof: To find the metric coefficients, we start with the following subset of equations of system (78):

$$\hat{L}_1(g_{11}) = 0 \quad \hat{L}_2(g_{11}) = -g_{13} \quad \hat{L}_3(g_{11}) = g_{12} \tag{32}$$

First, observe that $[\hat{L}_1, \hat{L}_2] = \hat{L}_3$ (see Appendix C). By application of Lemma 3.5, the following equation must hold:

$$-\hat{L}_1(g_{13}) = g_{12}. \tag{33}$$

But from (78), we have:

$$\hat{L}_1(g_{13}) = -\frac{1}{2} g_{12}.$$

Therefore, Eq. (33) becomes:

$$\frac{1}{2} g_{12} = g_{12}.$$

Obviously, this implies that $g_{12} = 0$. We next observe that $g_{12} = 0$ implies $\hat{L}_i(g_{12}) = 0, \quad i = 1, \ldots, 6$. From the system (78) we obtain:

$$\begin{array}{lll} g_{13} = 0 & g_{23} = 0 & g_{11} = g_{22} \\ g_{16} = 0 & g_{26} = 0 & g_{14} = g_{25} \end{array} \tag{34}$$

By using the equations for g_{13}, g_{23}, g_{16} and g_{26} from (78), and the fact that these coefficients are all 0, we further obtain

$$\begin{array}{lll} g_{15}=0 & g_{24}=0 & g_{11}=g_{33} \\ g_{34}=0 & g_{35}=0 & g_{14}=g_{36} \\ g_{44}=0 & g_{45}=0 & g_{46}=0 \\ g_{55}=0 & g_{56}=0 & g_{66}=0 \end{array} \tag{35}$$

Next observation is that $\hat{L}_i(g_{11})=0, \quad i=1,\ldots,6$. This, together with Eq. (34) and (35) implies:

$$g_{11}=g_{22}=g_{33}=\alpha.$$

where α is an arbitrary constant. Similarly, we obtain

$$g_{14}=g_{25}=g_{36}=\beta.$$

for an arbitrary constant β. In this way we have obtained all 21 independent values of G. The reader can easily check that all the equations (78) are satisfied by the above values so the theorem is proved.

■

Corollary 3.7 *There is no Riemannian metric whose geodesics are screw motions.*

Proof: It is easy to check that the matrix of the form

$$G=\begin{bmatrix} \alpha\, I_{3\times 3} & \beta\, I_{3\times 3} \\ \beta\, I_{3\times 3} & 0_{3\times 3} \end{bmatrix},$$

has two distinct real eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(\alpha+\sqrt{\alpha^2+4\beta^2}) \\ \lambda_2 &= \frac{1}{2}(\alpha-\sqrt{\alpha^2+4\beta^2}) \end{aligned}$$

which both have multiplicity 3. For any choice of α and β, the product of the eigenvalues is $\lambda_1\lambda_2=-4\beta^2\le 0$. Therefore, G is not positive-definite as required for a Riemannian metric.

■

Although G does not define a Riemannian metric, it induces a metrical structure on $SE(3)$ that can be treated in the analogous way as Riemannian structure. In particular, we can investigate the invariance properties of the metric given by G. By definition, the metric is left-invariant if for any $A, B \in SE(3)$ and for any vector fields X and Y:

$$< AX(B), AY(B) >_{AB}=< X(B), Y(B) >_B, \tag{36}$$

and it is right-invariant if:

$$< X(B)A, Y(B)A >_{BA}=< X(B), Y(B) >_B . \tag{37}$$

Before we investigate the invariance properties of the metric (31), we prove the following lemma:

Lemma 3.8 *If S_1 and S_2 are two elements of $se(3)$ and $SE(3)$ has the metric (31), then for any $A\in SE(3)$*

$$< S_1, S_2 >_I=< \mathrm{Ad}_A(S_1), \mathrm{Ad}_A(S_2) >_I . \tag{38}$$

If S is represented in its matrix form, the map $\mathrm{Ad}: se(3)\to se(3)$ is defined by $\mathrm{Ad}_A(S)=A\,S\,A^{-1}$.

Proof: Let $S_1=\{\omega_1, v_1\}$ and $S_2=\{\omega_2, v_2\}$. By a straightforward algebraic calculation it can be shown that for $S=\{\omega, v\}\in se(3)$ and $A\in SE(3)$, where

$$A=\begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

the value of $\mathrm{Ad}_A(S)$ is given by:

$$\mathrm{Ad}_A(S)=\{R\,\omega, R\,v-(R\omega)\times d\} \tag{39}$$

where $\times$ is the usual vector cross product. Therefore, we have:

$$\begin{aligned}
&< \mathrm{Ad}_A(S_1), \mathrm{Ad}_A(S_2) >_I = < \{R\,\omega_1, R\,v_1 - (R\omega_1)\times d\}, \{R\,\omega_2, R\,v_2 - (R\omega_2)\times d\} >_I \\
&\quad = \alpha\,(R\,\omega_1)^T(R\,\omega_2) + \beta\,(R\,\omega_1)^T(R\,v_2 - (R\omega_2)\times d) + \beta\,(R\,\omega_2)^T(v_1 - (R\omega_1)\times d) \\
&\quad = \alpha\,\omega_1^T\omega_2 + \beta\,(\omega_1^T v_2 + \omega_2^T v_1) - \beta((R\,\omega_1)^T((R\omega_2)\times d) + (R\,\omega_2)^T((R\omega_1)\times d)) \\
&\quad = \alpha\,\omega_1^T\omega_2 + \beta\,(\omega_1^T v_2 + \omega_2^T v_1) = < \{\omega_1, v_1\}, \{\omega_2, v_2\} >_I = < S_1, S_2 >_I
\end{aligned} \tag{40}$$

■

The lemma implies:

Proposition 3.9 *Metric G given by (31) is bi-invariant (both, left- and right-invariant).*

Proof: It is obvious that the metric G is left-invariant, since it is constant for the basis of the left-invariant vector fields $\hat{L}_i$. To show that it is also right invariant, take two vector fields X and Y. We have to check that Eq. (37) holds. First, since the metric is left-invariant, we have:

$$< X(B)A, Y(B)A >_{B\,A} = < (B\,A)^{-1}X(B)A, (B\,A)^{-1}Y(B)A >_I = < A^{-1}B^{-1}X(B)A, A^{-1}B^{-1}Y(B)A >_I .$$

By Lemma 3.8,

$$< A^{-1}B^{-1}X(B)A, A^{-1}B^{-1}Y(B)A >_I = < B^{-1}X(B), B^{-1}Y(B) >_I .$$

But because of the left-invariance of G, the last expression is:

$$< B^{-1}X(B), B^{-1}Y(B) >_I = < X(B), Y(B) >_B,$$

as required.

■

Analogously to the Riemannian case, we could define length of a curve $\gamma(t)$ between two points A_1 and A_2 on $SE(3)$ by:

$$L(\gamma; A_1, A_2) = \int_{A_1}^{A_2} < \frac{d\gamma}{dt}, \frac{d\gamma}{dt} >^{\frac{1}{2}} dt. \tag{41}$$

But G is not positive definite, so length of a curve would be in general a complex number. Instead, we define the energy measure:

$$E(\gamma; A_1, A_2) = \int_{A_1}^{A_2} < \frac{d\gamma}{dt}, \frac{d\gamma}{dt} > dt. \tag{42}$$

Since G is not positive definite, energy of a curve can be in general negative. There are also curves (other than points) which have zero energy.

Two special cases of metric (31) are particularly interesting. With $\alpha = 0$ and $\beta = 1$ we obtain the metric:

$$G = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{bmatrix}.$$

This metric, taken as a quadratic form on $se(3)$, is known as Klein form. The eigenvalues for the metric are $\{1, 1, 1, -1, -1, -1\}$ and the form is therefore non-degenerate. For a screw motion:

$$A(t) = A_0 \exp(t\,S)$$

the energy of the segment $t \in [0, 1]$ is given by

$$E(A) = \frac{2}{3}\omega^T v \tag{43}$$

where $S = \{\omega, v\} \in se(3)$. This implies that the energy of the geodesics will be positive when $\omega^T v > 0$, it will be negative when $\omega^T v < 0$ and it will be 0 if $\omega^T v = 0$. If $\omega \neq 0$, the quantity:

$$h = \frac{\omega^T v}{|\omega|^2} \tag{44}$$

is a constant known as a *pitch* of the screw motion. Physically, the pitch of the screw motion determines how much translation along the screw axis occurs per rotation about the axis. Zero energy screw motions with $\omega \neq 0$ are therefore pure rotations. On the other hand, if $\omega = 0$, the energy of the screw motion will be also 0. Screw motions with $\omega = 0$ are pure translations. In short, zero energy screw motions are either pure rotations or pure translations. Screw motions with positive energy are those with positive pitch. Trajectories for such motions correspond to right-handed spirals and the motions are also called right-handed screw motions. Analogously, screw motions with negative energy correspond to left-handed screw motions. Since pure rotations and pure translations are zero-energy motions, it is always possible to find a zero energy curve between two arbitrary points by breaking the motion into a segment consisting of pure rotation followed by a segment of pure translation. A natural choice of orthonormal basis on $se(3)$ for the Klein from are screw motions along the x, y and z axes with pitch $+1$ and -1. In kinematic literature, these motions are known as principal screws.

By putting $\alpha = 1$ and $\beta = 0$, we get the metric:

$$G = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

This metric, as a form on $se(3)$, is called the Killing form. Its eigenvalues are $\{1, 1, 1, 0, 0, 0\}$ so it is degenerate. The energy of a screw motion with $S = \{\omega, v\}$ is equal to $\frac{1}{3}\omega^T \omega$ so it is always non-negative. Pure translations are zero-energy motions while any motion involving rotation will have positive energy.

In the general case, for $\alpha \neq 0$ and $\beta \neq 0$, the energy of a screw motion along $S = \{\omega, v\}$ is $\frac{1}{3}\omega^T(\alpha\,\omega + 2\beta\,v)$. Zero-energy motions are pure translations. Whether the energy of the motion will be positive or negative in this case depends also on α and β, not only on the geometry.

4 Affine connections on SE(3)

Once a differentiable structure on a manifold is defined, notions of a tangent vector and a vector field naturally follow. To measure distances on the manifold we had to introduce a metric. This, in turn, allowed definition of a length of a curve and a geodesic as a curve which minimizes the distance between two points. There is no natural choice of metric – any $n \times n$ symmetric matrix G whose components are differentiable functions (in other words, any symmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor field) can be chosen. In the previous section, we chose a particular family of curves and found the metric G for which these curves were geodesics. In this process we interpreted geodesics as shortest distance curves thus notion of the distance was implicitly assumed.

In this section, we take a different approach: We try to introduce additional structure on the manifold until we have enough constraints that the metric eventually follows. We are guided by the notion of acceleration from kinematics. If a manifold is endowed with a metric, there is a natural way to differentiate vector fields, given by the metrical connection. Instead, we start by introducing a connection to $SE(3)$ so that the acceleration can be computed. By requiring that the acceleration computed through the connection agrees with the usual expression for the acceleration, we are able to partially characterize the affine connection. We then require that the connection is symmetric and show that this determines a unique connection. Finally, we find a class of metrics which are compatible with the derived affine connection.

4.1 A physically meaningful connection

At the beginning of the paper we have shown that the space of twists known from kinematics is isomorphic to the Lie algebra $se(3)$. Furthermore, by introducing the basis of left-invariant vector fields $\hat{L}_i$ (Eq. 8) we found a natural framework for studying motion on $SE(3)$: The components of tangent vector fields with respect to the basis $\hat{L}_i$ correspond to components of the instantaneous twist associated with the motion, expressed in the body-fixed coordinate frame. To arrive at the acceleration, we compute a covariant derivative of the velocity (that is, the tangent vector field) along the curve describing the motion.

We now turn our attention to the acceleration as computed in kinematics. Let $A(t)$ be a curve describing motion of a rigid body. Let $V(t) = \{\omega, v\}$ represent the instantaneous velocity of the rigid body, expressed in the moving frame M fixed to the rigid body. More precisely, ω represents the angular velocity of the rigid body while

v represents velocity of the origin of the body-fixed coordinate frame M, both expressed in the frame M. The acceleration of the rigid body is given by time derivative of $V(t)$ (where $V(t)$ is viewed simply as a vector in $\mathbb{R}^6$):

$$\frac{dV}{dt} = \left[\begin{array}{c} \dot{\omega} \\ \dot{v} \end{array} \right] + \left[\begin{array}{c} 0 \\ \omega \times v \end{array} \right]. \tag{45}$$

The second term in the acceleration occurs due to rotation of the frame M in which the vectors are expressed.

On the other hand, geometrically the acceleration of the rigid body, X, is given by the covariant derivative of $\hat{V}$ along $A(t)$:

$$X = \nabla_{\hat{V}} \hat{V} \tag{46}$$

Note that the twist $V(t)$ belongs to $se(3)$ for all t, while $\hat{V}(A_0)$ at $A_0 = A(t)$ belongs to the tangent space $T_{A_0}\, SE(3)$!

Let $X = X^i \hat{L}_i = \{\omega, v\}$. According to Eq. (45), the affine connection that produces kinematically consistent acceleration must satisfy:

$$\nabla_X X = \left[\begin{array}{c} \dot{\omega} \\ \dot{v} \end{array} \right] + \left[\begin{array}{c} 0_{3\times 1} \\ \omega \times v \end{array} \right]. \tag{47}$$

But in components, $\nabla_X X$ can be rewritten as:

$$\nabla_X X = \frac{dX^k}{dt} \hat{L}_k + X^i X^j \Gamma^k_{ji} \hat{L}_k. \tag{48}$$

where Γ^k_{ji} are the Christoffel symbols (which define the affine connection) for the basis $\hat{L}_i$. The two expressions (47) and (48) will be the same if the first and the second term in Eq. (47) correspond to the first and the second term in Eq. (48), respectively. Obviously, the first terms are the same regardless of the choice of the affine connection. However, because of the symmetry of the coefficients in the second term in Eq. (48), we can only conclude that:

$$\sum_k \sum_{j,i\leq j} X^i X^j (\Gamma^k_{ji} + \Gamma^k_{ij}) \hat{L}_k = \left[\begin{array}{c} 0_{3\times 1} \\ \omega \times v \end{array} \right]. \tag{49}$$

(The reader should be reminded that $\omega = \{X^1, X^2, X^3\}$ and $v = \{X^4, X^5, X^6\}$.) In this way we obtain a set of 126 equations of the following form:

$$\Gamma^k_{ij} + \Gamma^k_{ji} = a^k_{ij} \quad j = 1, \ldots, 6 \quad i \leq j \tag{50}$$

where a^k_{ij} are constants that can be directly obtained from the right-hand side of Eq. (49). The only non-zero values a^k_{ij} are:

$$\begin{array}{lll} a^6_{24} = -1 & a^5_{34} = 1 & a^6_{15} = 1 \\ a^4_{35} = -1 & a^5_{16} = -1 & a^4_{26} = 1 \end{array} \tag{51}$$

It is clear that the system (50) does not contain enough equations to solve for Γ^k_{ij} if $i \neq j$. However, the equations imply that $\Gamma^k_{ii} = 0$.

To obtain a unique solution for the remaining Christoffel symbols we have to impose additional constraints on the connection. One desirable property is that the covariant derivatives of coordinate basis vectors are symmetric:

$$\nabla_{\frac{\partial}{\partial \xi_i}} \frac{\partial}{\partial \xi_j} = \nabla_{\frac{\partial}{\partial \xi_j}} \frac{\partial}{\partial \xi_i}. \tag{52}$$

It is not difficult to check that this is exactly the definition of the symmetry from 2.5. Therefore, for general vector fields, symmetry of the connection is equivalent to:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{53}$$

It immediately follows that for the basis $\hat{L}_i$ the symmetry of the connection implies:

$$\Gamma^k_{ji} - \Gamma^k_{ij} = C^k_{ij}. \tag{54}$$

Equations (54) and (50) together uniquely specify the Christoffel symbols Γ^k_{ji} and therefore the connection. We will call this connection the *kinematic connection.* The non-zero Christoffel symbols for the kinematic connection are:

$$\begin{aligned} \Gamma^3_{21} = \Gamma^2_{13} = \Gamma^1_{32} = \frac{1}{2}, \qquad & \Gamma^3_{12} = \Gamma^2_{31} = \Gamma^1_{23} = -\frac{1}{2} \\ \Gamma^6_{51} = \Gamma^4_{62} = \Gamma^5_{43} = 1, \qquad & \Gamma^6_{42} = \Gamma^4_{53} = \Gamma^5_{61} = -1 \end{aligned} \tag{55}$$

4.2 Choice of metric that produces physically meaningful acceleration

Next, we answer the question whether there exists a metric for which the kinematic connection obtained in the previous subsection is the Riemannian connection. We show that there is a whole family of metrics for which this is true.

As seen in 2.5, a connection will be Riemannian if it is symmetric and compatible with the metric. Since we explicitly required that the kinematic connection be symmetric, we have to find a metric, which is compatible with the connection. Therefore, we must find a metric for which:

$$Z < X, Y > = < \nabla_Z X, Y > + < X, \nabla_Z Y >, \tag{56}$$

where X, Y and Z are arbitrary vector fields. By substituting the basis vector fields $\hat{L}_i$, $\hat{L}_j$ and $\hat{L}_k$ for X, Y and Z, the compatibility condition becomes:

$$\hat{L}_k(g_{ij}) = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li} \tag{57}$$

where the Christoffel symbols Γ^k_{ji} were computed above. Because of the symmetry of the metric coefficients, Eq. (57) produces a system of 126 partial differential equations for metric coefficients, similarly to Eq. (25) in Section 3.2. The complete set of equations is listed in Appendix B.

In finding the solution for the functions g_{ij} we initially proceed in the same way as in 3.2, using Lemma 3.5. Take the following subset of equations of (80):

$$\hat{L}_1(g_{11}) = 0 \quad \hat{L}_2(g_{11}) = -g_{13} \quad \hat{L}_3(g_{11}) = g_{12} \tag{58}$$

According to Lemma 3.5 the following equality holds:

$$-\hat{L}_1(g_{13}) = g_{12}.$$

By substituting for $\hat{L}_1(g_{13})$ from Eqs. (80), we obtain:

$$\frac{1}{2} g_{12} = g_{12},$$

which gives $g_{12} = 0$. Substituting in the system (80), we next obtain:

$$g_{13} = 0 \quad g_{23} = 0 \quad g_{11} = g_{22} \quad g_{11} = g_{33} \tag{59}$$

It is easy to see that these equations imply $\hat{L}_i(g_{11}) = 0, \quad i = 1, \ldots, 6$, which together with Eq. (59) results in:

$$g_{11} = g_{22} = g_{33} = \alpha,$$

where α is a constant. Therefore, the upper-left 3×3 block in the matrix G is of the form $\alpha I_{3\times3}$, where I is the identity matrix.

By taking equations:

$$\hat{L}_1(g_{14}) = 0 \quad \hat{L}_2(g_{14}) = -\tfrac{1}{2} g_{34} - g_{16} \quad \hat{L}_3(g_{14}) = \tfrac{1}{2} g_{24} + g_{15}, \tag{60}$$

and again using Lemma 3.5, we get $g_{24} = 0$. By substituting this in the system (80) it is easy to see that all the entries in the upper-right 3×3 block of the matrix G (and since G is symmetric, also of the lower-left 3×3 block) are equal to 0.

We can try to proceed in the same way with the rest of the equations for the entries of the lower-right 3×3 block. However, it turns out that the equations are consistent, in all cases Lemma 3.5 is trivially satisfied. Therefore, we have to use different approach to find the rest of the coefficients.

We begin with a simple result:

Proposition 4.1 *A family of left invariant metrics:*

$$G = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix}, \tag{61}$$

where α and β are arbitrary constants, satisfies all the equations (80). This are the only left-invariant metrics which are compatible with the acceleration connection.

Proof: If a metric is left invariant then the matrix G is constant. But if $g_{ij} = \text{const.}$, then $\hat{L}_i(g_{ij}) = 0$. It is then easy to check that the form of G in (61) follows from the equations (80).

■

To determine whether there are other solutions, it helps to investigate what do metrics which share the same metrical connection, have in common geometrically. A geometric entity that is determined from the connection is a geodesic (cf. Eq. 20). Therefore, two metrics with the same connection will have the same family of geodesics. By reasoning similar to the one that led to the acceleration connection we can prove that also the converse is true: The family of geodesics uniquely determines the symmetric connection. A manifold on which two different metrics are defined can be viewed as two different manifolds with the same differential structure. Thus, to study which metrics have the same geodesics, we study diffeomorphisms which map geodesics to geodesics. Such maps are called *affine maps*.

Park [10] and Žefran *et al.* [11] showed that the geodesics for metric (61) are the same as geodesics on the product manifold $SO(3) \times \mathbb{R}^3$ (which is the underlying topological space for $SE(3)$) endowed with the bi-invariant metric on $SO(3)$ and the Euclidean metric on $\mathbb{R}^3$. It can be actually shown that the metric (61) is a product metric itself [11]. Geometrically, the geodesic between two points A_1 and A_2 on $SE(3)$ is then a product of a geodesic on $SO(3)$ (basically a screw motion between two frames with the same origin) and a geodesic on $\mathbb{R}^3$ (a straight line). But on $\mathbb{R}^3$, straight lines are geodesics for an arbitrary inner product (defined by a positive-definite constant matrix). Therefore, any product metric on $SO(3) \times \mathbb{R}^3$ with the bi-invariant metric on $SO(3)$ and an inner product metric on $\mathbb{R}^3$ will also have the same geodesics as metric (61). If the basis $\hat{L}_1$, $\hat{L}_2$ and $\hat{L}_3$ is chosen for the vector fields on $SO(3)$ and the Euclidean basis E_4, E_5, E_6 for the vector fields on $\mathbb{R}^3$, the product metric G_p for this basis has the form:

$$G_p = \begin{bmatrix} \alpha I_{3\times 3} & 0 \\ 0 & W \end{bmatrix} \tag{62}$$

where W is a constant 3×3 positive-definite matrix defining an inner product on $\mathbb{R}^3$. Change of the basis vector fields on $\mathbb{R}^3$ will only change the lower-right block of matrix G_p. To compare metric (62) with (61) we have to change the basis on $\mathbb{R}^3$ from E_4, E_5, E_6 to $\hat{L}_4$, $\hat{L}_5$ and $\hat{L}_6$.

Take a point $A \in SE(3)$, where:

$$A = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}.$$

Note that A, as element of $SO(3) \times \mathbb{R}^3$ is represented by a pair (R, d). Take a vector field:

$$X = X^4 \hat{L}_4 + X^5 \hat{L}_5 + X^6 \hat{L}_6 = v^T \begin{bmatrix} \hat{L}_4 \\ \hat{L}_5 \\ \hat{L}_6 \end{bmatrix}, \tag{63}$$

where the components X^4, X^5 and X^6 are constants and $v = \{X^4, X^5, X^6\}^T$. The integral curve of this vector field passing through A is given by $\gamma(t) = A\ \exp(tT_X)$, where T_X is the matrix representation of the vector $\{0, 0, 0, X^4, X^5, X^6\} \in se(3)$. It is easy to see that:

$$\gamma(t) = \begin{bmatrix} R & tR\,v + d \\ 0 & 1 \end{bmatrix}.$$

The tangent vector to a curve $\gamma(t) = \{\xi_1(t), \ldots, \xi_6(t)\}$ is given by:

$$\frac{d\gamma}{dt} = \frac{d\xi_i}{dt}\frac{\partial}{\partial \xi_i}$$

The expression for the tangent vector $X(A)$ in the basis $E_i = \frac{\partial}{\partial \xi_i}$ is therefore:

$$v^T \begin{bmatrix} \hat{L}_4 \\ \hat{L}_5 \\ \hat{L}_6 \end{bmatrix} = (R\, v)^T \begin{bmatrix} E_4 \\ E_5 \\ E_6 \end{bmatrix} \tag{64}$$

Since the equation must be true for arbitrary v, we get:

$$\begin{bmatrix} \hat{L}_4 \\ \hat{L}_5 \\ \hat{L}_6 \end{bmatrix} = R^T \begin{bmatrix} E_4 \\ E_5 \\ E_6 \end{bmatrix} \tag{65}$$

The elements of the matrix G for the basis $\hat{L}_i$ are:

$$G = \begin{bmatrix} \hat{L}_4 \\ \hat{L}_5 \\ \hat{L}_6 \end{bmatrix} \cdot \begin{bmatrix} \hat{L}_4 & \hat{L}_5 & \hat{L}_6 \end{bmatrix} = R^T \begin{bmatrix} E_4 \\ E_5 \\ E_6 \end{bmatrix} \cdot \begin{bmatrix} E_4 & E_5 & E_6 \end{bmatrix} R = R^T W R, \tag{66}$$

where $E_i \cdot E_j \stackrel{\text{def}}{=} < E_i, E_j >$.

We have therefore arrived at the result that any metric described by a matrix:

$$G(R, d) = \begin{bmatrix} \alpha I & 0 \\ 0 & R^T\, W\, R \end{bmatrix} \tag{67}$$

where W is a positive definite matrix, is a solution to system of partial differential equations (80). Note that the form of G agrees with the previous analysis of the system (80). Another important observation is that for $W = \beta I$, (67) agrees with (61).

Equation (67) describes a whole family of metrics that are compatible with the acceleration connection. It turns out that there are no other metrics with these property – any metric that agrees with the acceleration connection has the form (67). To see this we concentrate on the system of partial differential equations (80) again. More precisely, we study the equations describing the lower-right 3×3 block of the matrix G. Denote this 3×3 symmetric matrix by G_T:

$$G_T = \begin{bmatrix} g_{44} & g_{45} & g_{46} \\ g_{45} & g_{55} & g_{56} \\ g_{46} & g_{56} & g_{66} \end{bmatrix}.$$

Proposition 4.2 *Let I_{G_T}, II_{G_T} and III_{G_T} be the three invariants of the matrix G. That is:*

$$\begin{aligned} I_{G_T} &= \mathrm{Trace}(G_T) = g_{44} + g_{55} + g_{66} \\ II_{G_T} &= g_{45}^2 + g_{46}^2 + g_{56}^2 - g_{44}g_{55} - g_{44}g_{66} - g_{55}g_{66} \\ III_{G_T} &= \mathrm{Det}(G_T) = g_{44}g_{55}g_{66} + 2g_{45}g_{46}g_{56} - g_{44}g_{56}^2 - g_{55}g_{46}^2 - g_{66}g_{45}^2 \end{aligned} \tag{68}$$

These matrix invariants are first integrals for the system of partial differential equations (80).

Proof: The proof is a pure algebraic manipulation: Function Φ is a first integral of the system (80) if and only if $\hat{L}_i(\Phi) = 0$ for $i = 1, \ldots, 6$. Using expressions for $\hat{L}_i(g_{ij})$ from (80) and the Leibniz' rule $\hat{L}_i(f\, g) = \hat{L}_i(f)\, g + f\, \hat{L}_i(g)$ it is easy to check the claim.

■

Corollary 4.3 *If a matrix G_T represents a solution of (80), then it can be decomposed in the following form:*

$$G_T = U\, \Lambda\, U^T, \tag{69}$$

where $U \in SO(3)$ is an orthogonal matrix which varies over the manifold and Λ is a constant (nonsingular) diagonal matrix.

Proof: The invariants of the matrix correspond to the coefficients of its characteristic polynomial. Since they are constant, the eigenvalues (which are the roots of the characteristic polynomial) of G_T will be constant. The decomposition (69) is a simple fact from the linear algebra.

■

Next, we state the second result of our paper:

Theorem 4.4 *A metric $G(R, d)$ is compatible with the acceleration connection given by Eq. (49) and (54) if and only if it has the form:*

$$G(R,d) = \begin{bmatrix} \alpha I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & R^T\,W\,R \end{bmatrix} \tag{70}$$

where W is a positive definite constant matrix.

Proof: We have seen that if G has the form (70) then it will be compatible with the acceleration connection. This proves the ($\Leftarrow$) direction of the theorem.

We have showed that if G is compatible with the acceleration connection (that is, it solves the system (80)) it must have the form:

$$G = \begin{bmatrix} \alpha I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & G_T \end{bmatrix}.$$

To complete the proof for the ($\Rightarrow$) direction we therefore have to show that $G_T(R, d) = R^T\,W\,R$ where W is a constant positive-definite matrix.

Assume that G_T is a solution of (80). According to (69), G_T can be written as:

$$G_T(R,d) = U\,\Lambda\,U^T$$

where $U = U(R, d) \in SO(3)$. Let

$$V(R,d) = R\,U$$

and

$$W = V\,\Lambda\,V^T \tag{71}$$

Obviously,

$$U\,\Lambda\,U^T = R^T\,W\,R,$$

and therefore:

$$G_T = R^T\,W\,R.$$

Let the part of the system (80) which describes G_T be written in the form

$$\begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} G_T = M, \tag{72}$$

where M is a 3×9 matrix of the right-hand sides of the equations in (80) which only depends on the entries of G_T. Fix a point $(R, d) \in SE(3)$ and set W' to be a constant matrix $W' = W(R, d)$. Define a matrix $G'_T = R^T\,W'\,R$. At (R, d), $G_T = G'_T$. But G'_T is a solution of (80), according to the first part of the proof. Therefore,

$$\begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} G'_T = M'$$

We have assumed that G_T is also a solution of (80). This means that:

$$\begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} G_T = M.$$

The matrices M and M' depend only on entries of G_T and G'_T, respectively. Since at (R, d), $G_T = G'_T$, it follows that at (R, d):

$$M = M'. \tag{73}$$

At (R, d), we also have:

$$M = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} G_T = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} G'_T + \begin{bmatrix} R\,L_1(W)\,R^T & R\,L_2(W)\,R^T & R\,L_3(W)\,R^T \end{bmatrix}. \tag{74}$$

Equations (73) and (74) together imply:

$$M = M + \left[\begin{array}{ccc} R\,L_1(W)\,R^T & R\,L_2(W)\,R^T & R\,L_3(W)\,R^T \end{array} \right]. \tag{75}$$

¿From the last equation it clearly follows that:

$$\left[\begin{array}{ccc} L_1 & L_2 & L_3 \end{array} \right] W = 0_{3\times 9}$$

which finally implies:

$$W = \text{const.} \tag{76}$$

To complete the proof, we observe that Eq. (71) implies that W is positive-definite.

■

5 Conclusion

Spatial displacements form a Lie group $SE(3)$. The Lie algebra of $SE(3)$, denoted by $se(3)$, is isomorphic to the space of twists and therefore provides a natural setting for analysis of instantaneous motions. However, in order to engage in higher order kinematic analysis or talk about the length of curves on the manifold, the differentiable structure of $SE(3)$ is not enough: To be able to differentiate a notion of covariant derivative must be introduced, while a Riemannian metric must be defined to define the length of a curve.

In this paper we investigate how additional structure can be introduced to $SE(3)$ to obtain notions that are familiar from the kinematics literature. First, we show that a natural setting to study screw motions is $SE(3)$ equipped with a two-parameter family of semi-Riemannian metrics. These metrics are non-definite and in general they are non-degenerate. Viewed as a quadratic form on $se(3)$, the metrics are a sum of the Killing form and the Klein form. When a metric in this family is non-degenerate, it defines a unique symmetric connection and screw motions are unique geodesics. When the metric is degenerate, as a form on $se(3)$ it is a scalar multiple of the Killing form. In this case the symmetric connection compatible with the metric is not unique any more and geodesics are not unique. However, screw motions are still one of possible solutions for geodesics.

To study acceleration or higher order derivatives of the velocity, $SE(3)$ must be equipped with an affine connection. The choice of the affine connection is restricted if we want to obtain the acceleration as known from physics. Further, if we require that the connection is symmetric, the connection is uniquely specified. In this case, a family of metrics which are compatible with this connection can be identified. All of them are product metrics: As a Riemannian manifold, $SE(3)$ is a Cartesian product of $SO(3)$ with the bi-invariant metric and $\mathbb{R}^3$ with the inner product metric. Alternatively, the symmetric connection studied in the paper can be viewed as a symmetric part of a general connection with non-zero torsion (asymmetric part). Since geodesics do not depend on torsion, the family of metrics compatible with the symmetric part of the connection corresponds to the metrics which have the geodesics given by this general connection.

A Equations defining metric with screw motions as geodesics

In Section 3.2 we concluded that Eq.(25) must be satisfied by the metric if screw motions are geodesics:

$$\hat{L}_k(g_{ij}) = \frac{1}{2}(C^l_{ki}g_{lj} + C^l_{kj}g_{li}). \tag{77}$$

Coefficients C^k_{ij} are the structure constants of the Lie algebra $se(3)$. We evaluated this equation in Mathematica to obtain a system of 126 partial differential equations, that have to be solved for the metric coefficients g_{ij}:

$$\begin{array}{lll}
\hat{L}_1(g_{11}) = 0 & \hat{L}_2(g_{11}) = -g_{13} & \hat{L}_3(g_{11}) = g_{12} \\
\hat{L}_4(g_{11}) = 0 & \hat{L}_5(g_{11}) = -g_{16} & \hat{L}_6(g_{11}) = g_{15} \\
\\
\hat{L}_1(g_{12}) = \frac{1}{2}g_{13} & \hat{L}_2(g_{12}) = -\frac{1}{2}g_{23} & \hat{L}_3(g_{12}) = \frac{1}{2}(-g_{11} + g_{22}) \\
\hat{L}_4(g_{12}) = \frac{1}{2}g_{16} & \hat{L}_5(g_{12}) = -\frac{1}{2}g_{26} & \hat{L}_6(g_{12}) = \frac{1}{2}(-g_{14} + g_{25}) \\
\\
\hat{L}_1(g_{13}) = -\frac{1}{2}g_{12} & \hat{L}_2(g_{13}) = \frac{1}{2}(g_{11} - g_{33}) & \hat{L}_3(g_{13}) = \frac{1}{2}g_{23} \\
\hat{L}_4(g_{13}) = -\frac{1}{2}g_{15} & \hat{L}_5(g_{13}) = \frac{1}{2}(g_{14} - g_{36}) & \hat{L}_6(g_{13}) = \frac{1}{2}g_{35} \\
\\
\hat{L}_1(g_{14}) = 0 & \hat{L}_2(g_{14}) = \frac{1}{2}(-g_{34} - g_{16}) & \hat{L}_3(g_{14}) = \frac{1}{2}(g_{24} + g_{15}) \\
\hat{L}_4(g_{14}) = 0 & \hat{L}_5(g_{14}) = -\frac{1}{2}g_{46} & \hat{L}_6(g_{14}) = \frac{1}{2}g_{45} \\
\\
\hat{L}_1(g_{15}) = \frac{1}{2}g_{16} & \hat{L}_2(g_{15}) = -\frac{1}{2}g_{35} & \hat{L}_3(g_{15}) = \frac{1}{2}(g_{25} - g_{14}) \\
\hat{L}_4(g_{15}) = 0 & \hat{L}_5(g_{15}) = -\frac{1}{2}g_{56} & \hat{L}_6(g_{15}) = \frac{1}{2}g_{55} \\
\\
\hat{L}_1(g_{16}) = -\frac{1}{2}g_{15} & \hat{L}_2(g_{16}) = \frac{1}{2}(-g_{36} + g_{14}) & \hat{L}_3(g_{16}) = \frac{1}{2}g_{26} \\
\hat{L}_4(g_{16}) = 0 & \hat{L}_5(g_{16}) = -\frac{1}{2}g_{66} & \hat{L}_6(g_{16}) = \frac{1}{2}g_{56} \\
\\
\hat{L}_1(g_{22}) = g_{23} & \hat{L}_2(g_{22}) = 0 & \hat{L}_3(g_{22}) = -g_{12} \\
\hat{L}_4(g_{22}) = g_{26} & \hat{L}_5(g_{22}) = 0 & \hat{L}_6(g_{22}) = -g_{24} \\
\\
\hat{L}_1(g_{23}) = \frac{1}{2}(-g_{22} + g_{33}) & \hat{L}_2(g_{23}) = \frac{1}{2}g_{12} & \hat{L}_3(g_{23}) = -\frac{1}{2}g_{13} \\
\hat{L}_4(g_{23}) = \frac{1}{2}(-g_{25} + g_{36}) & \hat{L}_5(g_{23}) = \frac{1}{2}g_{24} & \hat{L}_6(g_{23}) = -\frac{1}{2}g_{34} \\
\\
\hat{L}_1(g_{24}) = \frac{1}{2}g_{34} & \hat{L}_2(g_{24}) = -\frac{1}{2}g_{26} & \hat{L}_3(g_{24}) = \frac{1}{2}(-g_{14} + g_{25}) \\
\hat{L}_4(g_{24}) = \frac{1}{2}g_{46} & \hat{L}_5(g_{24}) = 0 & \hat{L}_6(g_{24}) = -\frac{1}{2}g_{44} \\
\\
\hat{L}_1(g_{25}) = \frac{1}{2}(g_{35} + g_{26}) & \hat{L}_2(g_{25}) = 0 & \hat{L}_3(g_{25}) = \frac{1}{2}(-g_{15} - g_{24}) \\
\hat{L}_4(g_{25}) = \frac{1}{2}g_{56} & \hat{L}_5(g_{25}) = 0 & \hat{L}_6(g_{25}) = -\frac{1}{2}g_{45} \\
\\
\hat{L}_1(g_{26}) = \frac{1}{2}(g_{36} - g_{25}) & \hat{L}_2(g_{26}) = \frac{1}{2}g_{24} & \hat{L}_3(g_{26}) = -\frac{1}{2}g_{16} \\
\hat{L}_4(g_{26}) = \frac{1}{2}g_{66} & \hat{L}_5(g_{26}) = 0 & \hat{L}_6(g_{26}) = -\frac{1}{2}g_{46}
\end{array} \tag{78}$$

$\hat{L}_1(g_{33}) = -g_{23}$ $\hat{L}_2(g_{33}) = g_{13}$ $\hat{L}_3(g_{33}) = 0$
$\hat{L}_4(g_{33}) = -g_{35}$ $\hat{L}_5(g_{33}) = g_{34}$ $\hat{L}_6(g_{33}) = 0$

$\hat{L}_1(g_{34}) = -\frac{1}{2}g_{24}$ $\hat{L}_2(g_{34}) = \frac{1}{2}(g_{14} - g_{36})$ $\hat{L}_3(g_{34}) = \frac{1}{2}g_{35}$
$\hat{L}_4(g_{34}) = -\frac{1}{2}g_{45}$ $\hat{L}_5(g_{34}) = \frac{1}{2}g_{44}$ $\hat{L}_6(g_{34}) = 0$

$\hat{L}_1(g_{35}) = \frac{1}{2}(-g_{25} + g_{36})$ $\hat{L}_2(g_{35}) = \frac{1}{2}g_{15}$ $\hat{L}_3(g_{35}) = -\frac{1}{2}g_{34}$
$\hat{L}_4(g_{35}) = -\frac{1}{2}g_{55}$ $\hat{L}_5(g_{35}) = \frac{1}{2}g_{45}$ $\hat{L}_6(g_{35}) = 0$

$\hat{L}_1(g_{36}) = \frac{1}{2}(-g_{26} - g_{35})$ $\hat{L}_2(g_{36}) = \frac{1}{2}(g_{16} + g_{34})$ $\hat{L}_3(g_{36}) = 0$
$\hat{L}_4(g_{36}) = -\frac{1}{2}g_{56}$ $\hat{L}_5(g_{36}) = \frac{1}{2}g_{46}$ $\hat{L}_6(g_{36}) = 0$

$\hat{L}_1(g_{44}) = 0$ $\hat{L}_2(g_{44}) = -g_{46}$ $\hat{L}_3(g_{44}) = g_{45}$
$\hat{L}_4(g_{44}) = 0$ $\hat{L}_5(g_{44}) = 0$ $\hat{L}_6(g_{44}) = 0$

$\hat{L}_1(g_{45}) = \frac{1}{2}g_{46}$ $\hat{L}_2(g_{45}) = -\frac{1}{2}g_{56}$ $\hat{L}_3(g_{45}) = \frac{1}{2}(-g_{44} + g_{55})$
$\hat{L}_4(g_{45}) = 0$ $\hat{L}_5(g_{45}) = 0$ $\hat{L}_6(g_{45}) = 0$

$\hat{L}_1(g_{46}) = -\frac{1}{2}g_{45}$ $\hat{L}_2(g_{46}) = \frac{1}{2}(g_{44} - g_{66})$ $\hat{L}_3(g_{46}) = \frac{1}{2}g_{56}$
$\hat{L}_4(g_{46}) = 0$ $\hat{L}_5(g_{46}) = 0$ $\hat{L}_6(g_{46}) = 0$

$\hat{L}_1(g_{55}) = g_{56}$ $\hat{L}_2(g_{55}) = 0$ $\hat{L}_3(g_{55}) = -g_{45}$
$\hat{L}_4(g_{55}) = 0$ $\hat{L}_5(g_{55}) = 0$ $\hat{L}_6(g_{55}) = 0$

$\hat{L}_1(g_{56}) = \frac{1}{2}(-g_{55} + g_{66})$ $\hat{L}_2(g_{56}) = \frac{1}{2}g_{45}$ $\hat{L}_3(g_{56}) = -\frac{1}{2}g_{46}$
$\hat{L}_4(g_{56}) = 0$ $\hat{L}_5(g_{56}) = 0$ $\hat{L}_6(g_{56}) = 0$

$\hat{L}_1(g_{66}) = -g_{56}$ $\hat{L}_2(g_{66}) = g_{46}$ $\hat{L}_3(g_{66}) = 0$
$\hat{L}_4(g_{66}) = 0$ $\hat{L}_5(g_{66}) = 0$ $\hat{L}_6(g_{66}) = 0$

B Equations for the metric compatible with the acceleration connection

In Section 4 we concluded that a metric compatible with the acceleration connection must satisfy:

$$\hat{L}_k(g_{ij}) = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li}. \tag{79}$$

The Christoffel symbols Γ^k_{ij} specify the acceleration connection and are listed in (55). The equation was expanded in Mathematica to obtain a system of 126 partial differential equations for the metric coefficients g_{ij}:

$$\begin{array}{lll}
\hat{L}_1(g_{11}) = 0 & \hat{L}_2(g_{11}) = -g_{13} & \hat{L}_3(g_{11}) = g_{12} \\
\hat{L}_4(g_{11}) = 0 & \hat{L}_5(g_{11}) = 0 & \hat{L}_6(g_{11}) = 0 \\[1em]
\hat{L}_1(g_{12}) = \frac{1}{2}g_{13} & \hat{L}_2(g_{12}) = -\frac{1}{2}g_{23} & \hat{L}_3(g_{12}) = -\frac{1}{2}g_{11} + \frac{1}{2}g_{22} \\
\hat{L}_4(g_{12}) = 0 & \hat{L}_5(g_{12}) = 0 & \hat{L}_6(g_{12}) = 0 \\[1em]
\hat{L}_1(g_{13}) = -\frac{1}{2}g_{12} & \hat{L}_2(g_{13}) = \frac{1}{2}g_{11} - \frac{1}{2}g_{33} & \hat{L}_3(g_{13}) = \frac{1}{2}g_{23} \\
\hat{L}_4(g_{13}) = 0 & \hat{L}_5(g_{13}) = 0 & \hat{L}_6(g_{13}) = 0 \\[1em]
\hat{L}_1(g_{14}) = 0 & \hat{L}_2(g_{14}) = -\frac{1}{2}g_{34} - g_{16} & \hat{L}_3(g_{14}) = \frac{1}{2}g_{24} + g_{15} \\
\hat{L}_4(g_{14}) = 0 & \hat{L}_5(g_{14}) = 0 & \hat{L}_6(g_{14}) = 0 \\[1em]
\hat{L}_1(g_{15}) = g_{16} & \hat{L}_2(g_{15}) = -\frac{1}{2}g_{35} & \hat{L}_3(g_{15}) = \frac{1}{2}g_{25} - g_{14} \\
\hat{L}_4(g_{15}) = 0 & \hat{L}_5(g_{15}) = 0 & \hat{L}_6(g_{15}) = 0 \\[1em]
\hat{L}_1(g_{16}) = -g_{15} & \hat{L}_2(g_{16}) = -\frac{1}{2}g_{36} + g_{14} & \hat{L}_3(g_{16}) = \frac{1}{2}g_{26} \\
\hat{L}_4(g_{16}) = 0 & \hat{L}_5(g_{16}) = 0 & \hat{L}_6(g_{16}) = 0 \\[1em]
\hat{L}_1(g_{22}) = g_{23} & \hat{L}_2(g_{22}) = 0 & \hat{L}_3(g_{22}) = -g_{12} \\
\hat{L}_4(g_{22}) = 0 & \hat{L}_5(g_{22}) = 0 & \hat{L}_6(g_{22}) = 0 \\[1em]
\hat{L}_1(g_{23}) = -\frac{1}{2}g_{22} + \frac{1}{2}g_{33} & \hat{L}_2(g_{23}) = \frac{1}{2}g_{12} & \hat{L}_3(g_{23}) = -\frac{1}{2}g_{13} \\
\hat{L}_4(g_{23}) = 0 & \hat{L}_5(g_{23}) = 0 & \hat{L}_6(g_{23}) = 0 \\[1em]
\hat{L}_1(g_{24}) = \frac{1}{2}g_{34} & \hat{L}_2(g_{24}) = -g_{26} & \hat{L}_3(g_{24}) = -\frac{1}{2}g_{14} + g_{25} \\
\hat{L}_4(g_{24}) = 0 & \hat{L}_5(g_{24}) = 0 & \hat{L}_6(g_{24}) = 0 \\[1em]
\hat{L}_1(g_{25}) = \frac{1}{2}g_{35} + g_{26} & \hat{L}_2(g_{25}) = 0 & \hat{L}_3(g_{25}) = -\frac{1}{2}g_{15} - g_{24} \\
\hat{L}_4(g_{25}) = 0 & \hat{L}_5(g_{25}) = 0 & \hat{L}_6(g_{25}) = 0 \\[1em]
\hat{L}_1(g_{26}) = \frac{1}{2}g_{36} - g_{25} & \hat{L}_2(g_{26}) = g_{24} & \hat{L}_3(g_{26}) = -\frac{1}{2}g_{16} \\
\hat{L}_4(g_{26}) = 0 & \hat{L}_5(g_{26}) = 0 & \hat{L}_6(g_{26}) = 0
\end{array} \tag{80}$$

$\hat{L}_1(g_{33}) = -g_{23}$	$\hat{L}_2(g_{33}) = g_{13}$	$\hat{L}_3(g_{33}) = 0$
$\hat{L}_4(g_{33}) = 0$	$\hat{L}_5(g_{33}) = 0$	$\hat{L}_6(g_{33}) = 0$
$\hat{L}_1(g_{34}) = -\frac{1}{2}g_{24}$	$\hat{L}_2(g_{34}) = \frac{1}{2}g_{14} - g_{36}$	$\hat{L}_3(g_{34}) = g_{35}$
$\hat{L}_4(g_{34}) = 0$	$\hat{L}_5(g_{34}) = 0$	$\hat{L}_6(g_{34}) = 0$
$\hat{L}_1(g_{35}) = -\frac{1}{2}g_{25} + g_{36}$	$\hat{L}_2(g_{35}) = \frac{1}{2}g_{15}$	$\hat{L}_3(g_{35}) = -g_{34}$
$\hat{L}_4(g_{35}) = 0$	$\hat{L}_5(g_{35}) = 0$	$\hat{L}_6(g_{35}) = 0$
$\hat{L}_1(g_{36}) = -\frac{1}{2}g_{26} - g_{35}$	$\hat{L}_2(g_{36}) = \frac{1}{2}g_{16} + g_{34}$	$\hat{L}_3(g_{36}) = 0$
$\hat{L}_4(g_{36}) = 0$	$\hat{L}_5(g_{36}) = 0$	$\hat{L}_6(g_{36}) = 0$
$\hat{L}_1(g_{44}) = 0$	$\hat{L}_2(g_{44}) = -2g_{46}$	$\hat{L}_3(g_{44}) = 2g_{45}$
$\hat{L}_4(g_{44}) = 0$	$\hat{L}_5(g_{44}) = 0$	$\hat{L}_6(g_{44}) = 0$
$\hat{L}_1(g_{45}) = g_{46}$	$\hat{L}_2(g_{45}) = -g_{56}$	$\hat{L}_3(g_{45}) = -g_{44} + g_{55}$
$\hat{L}_4(g_{45}) = 0$	$\hat{L}_5(g_{45}) = 0$	$\hat{L}_6(g_{45}) = 0$
$\hat{L}_1(g_{46}) = -g_{45}$	$\hat{L}_2(g_{46}) = g_{44} - g_{66}$	$\hat{L}_3(g_{46}) = g_{56}$
$\hat{L}_4(g_{46}) = 0$	$\hat{L}_5(g_{46}) = 0$	$\hat{L}_6(g_{46}) = 0$
$\hat{L}_1(g_{55}) = 2g_{56}$	$\hat{L}_2(g_{55}) = 0$	$\hat{L}_3(g_{55}) = -2g_{45}$
$\hat{L}_4(g_{55}) = 0$	$\hat{L}_5(g_{55}) = 0$	$\hat{L}_6(g_{55}) = 0$
$\hat{L}_1(g_{56}) = -g_{55} + g_{66}$	$\hat{L}_2(g_{56}) = g_{45}$	$\hat{L}_3(g_{56}) = -g_{46}$
$\hat{L}_4(g_{56}) = 0$	$\hat{L}_5(g_{56}) = 0$	$\hat{L}_6(g_{56}) = 0$
$\hat{L}_1(g_{66}) = -2g_{56}$	$\hat{L}_2(g_{66}) = 2g_{46}$	$\hat{L}_3(g_{66}) = 0$
$\hat{L}_4(g_{66}) = 0$	$\hat{L}_5(g_{66}) = 0$	$\hat{L}_6(g_{66}) = 0$

C Lie brackets for $se(3)$

In our derivations we need to evaluate Lie brackets of the basis vectors $\hat{L}_i$. Since these basis vectors are left invariant, it suffices to evaluate the brackets on $se(3)$ (see Eq. 9):

$$
\begin{array}{lll}
[L_1, L_1] = 0 & [L_1, L_2] = L_3 & [L_1, L_3] = -L_2 \\
[L_1, L_4] = 0 & [L_1, L_5] = L_6 & [L_1, L_6] = -L_5 \\
[L_2, L_2] = 0 & [L_2, L_3] = L_1 & [L_2, L_4] = -L_6 \\
[L_2, L_5] = 0 & [L_2, L_6] = L_4 & [L_3, L_3] = 0 \\
[L_3, L_4] = L_5 & [L_3, L_5] = -L_4 & [L_3, L_6] = 0 \\
[L_4, L_4] = 0 & [L_4, L_5] = 0 & [L_4, L_6] = 0 \\
[L_5, L_5] = 0 & [L_5, L_6] = 0 & [L_6, L_6] = 0
\end{array}
$$

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