

Lecture Notes

**An Introduction  
to  
Riemannian Geometry**

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## Preface

These lecture notes grew out of an M.Sc. course on differential geometry which I gave at the University of Leeds 1992. Their main purpose is to introduce the beautiful theory of Riemannian Geometry a still very active research area of mathematics. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work. Of special interest are the classical Lie groups allowing concrete calculations of many of the abstract notions on the menu.

The study of Riemannian Geometry is rather meaningless without some basic knowledge on Gaussian Geometry that is the differential geometry of curves and surfaces in 3-dimensional space. For this we recommend the excellent textbook: M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall (1976).

These lecture notes are written for students with a good understanding of linear algebra, real analysis of several variables, the classical theory of ordinary differential equations and some topology. The most important results stated in the text are also proved there. Other smaller ones are left to the reader as exercises, which follow at the end of each chapter. This format is aimed at students willing to put **hard work** into the course.

For further reading we recommend the very interesting textbook: M. P. do Carmo, *Riemannian Geometry*, Birkhäuser (1992).

I am very grateful to my many students who throughout the years have contributed to the text by finding numerous typing errors and giving many useful comments on the presentation.

It is my intention to extend this very incomplete draft version and include some of the differential geometry of the Riemannian symmetric spaces.

Lund University, 15 January 2004

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## CHAPTER 1

### **Introduction**

On the 10th of June 1854 Riemann gave his famous "Habilitationsvortrag" in the Colloquium of the Philosophical Faculty at Göttingen. His talk with the title "Über die Hypothesen, welche der Geometrie zu Grunde liegen" is often said to be the most important in the history of differential geometry. Gauss, at the age of 76, was in the audience and is said to have been very impressed by his former student.

Riemann's revolutionary ideas generalized the geometry of surfaces which had been studied earlier by Gauss, Bolyai and Lobachevsky. Later this led to an exact definition of the modern concept of an abstract Riemannian manifold.





## Differentiable Manifolds

The main purpose of this chapter is to introduce the concepts of a differentiable manifold, a submanifold and a differentiable map between manifolds. By this we generalize notions from the classical theory of curves and surfaces studied in most introductory courses on differential geometry.

For a natural number  $m$  let  $\mathbb{R}^m$  be the  $m$ -dimensional real vector space equipped with the topology induced by the standard Euclidean metric  $d$  on  $\mathbb{R}^m$  given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_m - y_m)^2}.$$

For positive natural numbers  $n, r$  and an open subset  $U$  of  $\mathbb{R}^m$  we shall by  $C^r(U, \mathbb{R}^n)$  denote the  $r$ -times continuously differentiable maps  $U \rightarrow \mathbb{R}^n$ . By **smooth** maps  $U \rightarrow \mathbb{R}^n$  we mean the elements of

$$C^\infty(U, \mathbb{R}^n) = \bigcap_{r=1}^{\infty} C^r(U, \mathbb{R}^n).$$

The set of **real analytic** maps  $U \rightarrow \mathbb{R}^n$  will be denoted by  $C^\omega(U, \mathbb{R}^n)$ . For the theory of real analytic maps we recommend the book: S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser (1992).

**Definition 2.1.** Let  $(M, \mathcal{T})$  be a topological Hausdorff space with a countable basis. Then  $M$  is said to be a **topological manifold** if there exists a natural number  $m$  and for each point  $p \in M$  an open neighbourhood  $U$  of  $p$  and a continuous map  $x : U \rightarrow \mathbb{R}^m$  which is a homeomorphism onto its image  $x(U)$  which is an open subset of  $\mathbb{R}^m$ . The pair  $(U, x)$  is called a **chart** (or **local coordinates**) on  $M$ . The natural number  $m$  is called the **dimension** of  $M$ . To denote that the dimension of  $M$  is  $m$  we write  $M^m$ .

Following Definition 2.1 a topological manifold  $M$  is locally homeomorphic to the standard  $\mathbb{R}^m$  for some natural number  $m$ . We shall now use the charts on  $M$  to define a differentiable structure and make  $M$  into a differentiable manifold.

**Definition 2.2.** Let  $M$  be a topological manifold. Then a  $C^r$ -atlas for  $M$  is a collection of charts

$$\mathcal{A} = \{(U_\alpha, x_\alpha) \mid \alpha \in I\}$$

such that  $\mathcal{A}$  covers the whole of  $M$  i.e.

$$M = \bigcup_{\alpha} U_\alpha$$

and for all  $\alpha, \beta \in I$  the corresponding **transition map**

$$x_\beta \circ x_\alpha^{-1}|_{x_\alpha(U_\alpha \cap U_\beta)} : x_\alpha(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^m$$

is  $r$ -times continuously differentiable.

A chart  $(U, x)$  on  $M$  is said to be **compatible** with a  $C^r$ -atlas  $\mathcal{A}$  on  $M$  if  $\mathcal{A} \cup \{(U, x)\}$  is a  $C^r$ -atlas. A  $C^r$ -atlas  $\hat{\mathcal{A}}$  is said to be **maximal** if it contains all the charts that are compatible with it. A maximal atlas  $\hat{\mathcal{A}}$  on  $M$  is also called a  $C^r$ -**structure** on  $M$ . The pair  $(M, \hat{\mathcal{A}})$  is said to be a  $C^r$ -**manifold**, or a **differentiable manifold** of class  $C^r$ , if  $M$  is a topological manifold and  $\hat{\mathcal{A}}$  is a  $C^r$ -structure on  $M$ . A differentiable manifold is said to be **smooth** if its transition maps are  $C^\infty$  and **real analytic** if they are  $C^\omega$ .

It should be noted that a given  $C^r$ -atlas  $\mathcal{A}$  on  $M$  determines a unique  $C^r$ -structure  $\hat{\mathcal{A}}$  on  $M$  containing  $\mathcal{A}$ . It simply consists of all charts compatible with  $\mathcal{A}$ . For the standard topological space  $(\mathbb{R}^m, \mathcal{T})$  we have the trivial  $C^\omega$ -atlas

$$\mathcal{A} = \{(\mathbb{R}^m, x) \mid x : p \mapsto p\}$$

inducing the standard  $C^\omega$ -structure  $\hat{\mathcal{A}}$  on  $\mathbb{R}^m$ .

**Example 2.3.** Let  $S^m$  denote the unit sphere in  $\mathbb{R}^{m+1}$  i.e.

$$S^m = \{p \in \mathbb{R}^{m+1} \mid p_1^2 + \cdots + p_{m+1}^2 = 1\}$$

equipped with the subset topology induced by the standard  $\mathcal{T}$  on  $\mathbb{R}^{m+1}$ . Let  $N$  be the north pole  $N = (1, 0) \in \mathbb{R} \times \mathbb{R}^m$  and  $S$  be the south pole  $S = (-1, 0)$  on  $S^m$ , respectively. Put  $U_N = S^m - \{N\}$ ,  $U_S = S^m - \{S\}$  and define  $x_N : U_N \rightarrow \mathbb{R}^m$ ,  $x_S : U_S \rightarrow \mathbb{R}^m$  by

$$x_N : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 - p_1}(p_2, \dots, p_{m+1}),$$

$$x_S : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 + p_1}(p_2, \dots, p_{m+1}).$$

Then the transition maps

$$x_S \circ x_N^{-1}, x_N \circ x_S^{-1} : \mathbb{R}^m - \{0\} \rightarrow \mathbb{R}^m - \{0\}$$

are given by

$$p \mapsto \frac{p}{|p|^2}$$

so  $\mathcal{A} = \{(U_N, x_N), (U_S, x_S)\}$  is a  $C^\omega$ -atlas on  $S^m$ . The  $C^\omega$ -manifold  $(S^m, \hat{\mathcal{A}})$  is called the standard  $m$ -dimensional **sphere**.

Another interesting example of a differentiable manifold is the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ .

**Example 2.4.** On the set  $\mathbb{R}^{m+1} - \{0\}$  we define the equivalence relation  $\equiv$  by  $p \equiv q$  if and only if there exists a  $\lambda \in \mathbb{R}^* = \mathbb{R} - \{0\}$  such that  $p = \lambda q$ . Let  $\mathbb{R}P^m$  be the quotient space  $(\mathbb{R}^{m+1} - \{0\})/\equiv$  and

$$\pi : \mathbb{R}^{m+1} - \{0\} \rightarrow \mathbb{R}P^m$$

be the natural projection, mapping a point  $p \in \mathbb{R}^{m+1} - \{0\}$  to the equivalence class  $[p] \in \mathbb{R}P^m$  containing  $p$ . Equip  $\mathbb{R}P^m$  with the quotient topology induced by  $\pi$  and  $\mathcal{T}$  on  $\mathbb{R}^{m+1}$ . For  $k \in \{1, \dots, m+1\}$  put

$$U_k = \{[p] \in \mathbb{R}P^m \mid p_k \neq 0\}$$

and define the charts  $x_k : U_k \rightarrow \mathbb{R}^m$  on  $\mathbb{R}P^m$  by

$$x_k : [p] \mapsto \left( \frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k} \right).$$

If  $[p] \equiv [q]$  then  $p = \lambda q$  for some  $\lambda \in \mathbb{R}^*$  so  $p_l/p_k = q_l/q_k$  for all  $l$ . This means that the map  $x_k$  is well defined for all  $k$ . The corresponding transition maps

$$x_k \circ x_l^{-1}|_{x_l(U_l \cap U_k)} : x_l(U_l \cap U_k) \rightarrow \mathbb{R}^m$$

are given by

$$\left( \frac{p_1}{p_l}, \dots, \frac{p_{l-1}}{p_l}, 1, \frac{p_{l+1}}{p_l}, \dots, \frac{p_{m+1}}{p_l} \right) \mapsto \left( \frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k} \right)$$

so the collection

$$\mathcal{A} = \{(U_k, x_k) \mid k = 1, \dots, m+1\}$$

is a  $C^\omega$ -atlas on  $\mathbb{R}P^m$ . The differentiable manifold  $(\mathbb{R}P^m, \hat{\mathcal{A}})$  is called the  $m$ -dimensional **real projective space**.

**Example 2.5.** Let  $\hat{\mathbb{C}}$  be the extended complex plane given by

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

and define  $\mathbb{C}^* = \mathbb{C} - \{0\}$ ,  $U_0 = \mathbb{C}$  and  $U_\infty = \hat{\mathbb{C}} - \{0\}$ . Then define the local coordinates  $x_0 : U_0 \rightarrow \mathbb{C}$  and  $x_\infty : U_\infty \rightarrow \mathbb{C}$  on  $\hat{\mathbb{C}}$  by  $x_0 : z \mapsto z$  and  $x_\infty : w \mapsto 1/w$ , respectively. The corresponding transition maps

$$x_\infty \circ x_0^{-1}, x_0 \circ x_\infty^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^*$$

are both given by  $z \mapsto 1/z$  so  $\mathcal{A} = \{(U_0, x_0), (U_\infty, x_\infty)\}$  is a  $C^\omega$ -atlas on  $\hat{\mathbb{C}}$ . The real analytic manifold  $(\hat{\mathbb{C}}, \hat{\mathcal{A}})$  is called the **Riemann sphere**.

For the product of two differentiable manifolds we have the following interesting result.

**Proposition 2.6.** *Let  $(M_1, \hat{\mathcal{A}}_1)$  and  $(M_2, \hat{\mathcal{A}}_2)$  be two differentiable manifolds of class  $C^r$ . Let  $M = M_1 \times M_2$  be the product space with the product topology. Then there exists an atlas  $\mathcal{A}$  on  $M$  making  $(M, \hat{\mathcal{A}})$  into a differentiable manifold of class  $C^r$  and the dimension of  $M$  satisfies*

$$\dim M = \dim M_1 + \dim M_2.$$

PROOF. See Exercise 2.1. □

The concept of a submanifold of a given differentiable manifold will play an important role as we go along and we shall be especially interested in the connection between the geometry of a submanifold and that of its ambient space.

**Definition 2.7.** Let  $m, n$  be natural numbers such that  $n \geq m$ ,  $n \geq 1$  and  $(N^n, \hat{\mathcal{B}})$  be a  $C^r$ -manifold. A subset  $M$  of  $N$  is said to be a **submanifold** of  $N$  if for each point  $p \in M$  there exists a chart  $(U_p, x_p) \in \hat{\mathcal{B}}$  such that  $p \in U_p$  and  $x_p : U_p \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}$  satisfies

$$x_p(U_p \cap M) = x_p(U_p) \cap (\mathbb{R}^m \times \{0\}).$$

The natural number  $(n - m)$  is called the **codimension** of  $M$  in  $N$ .

**Proposition 2.8.** *Let  $m, n$  be natural numbers such that  $n \geq m$ ,  $n \geq 1$  and  $(N^n, \hat{\mathcal{B}})$  be a  $C^r$ -manifold. Let  $M$  be a submanifold of  $N$  equipped with the subset topology and  $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$  be the natural projection onto the first factor. Then*

$$\mathcal{A} = \{(U_p \cap M, (\pi \circ x_p)|_{U_p \cap M}) \mid p \in M\}$$

is a  $C^r$ -atlas for  $M$ . In particular, the pair  $(M, \hat{\mathcal{A}})$  is an  $m$ -dimensional  $C^r$ -manifold. The differentiable structure  $\hat{\mathcal{A}}$  on the submanifold  $M$  of  $N$  is called the **induced structure** of  $\mathcal{B}$ .

PROOF. See Exercise 2.2. □

Our next step is to prove the implicit function theorem which is a useful tool for constructing submanifolds of  $\mathbb{R}^n$ . For this we use the classical inverse function theorem stated below. Note that if  $F : U \rightarrow$

$\mathbb{R}^m$  is a differentiable map defined on an open subset  $U$  of  $\mathbb{R}^n$  then its derivative  $DF_p$  at a point  $p \in U$  is the  $m \times n$  matrix given by

$$DF_p = \begin{pmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m/\partial x_1(p) & \dots & \partial F_m/\partial x_n(p) \end{pmatrix}.$$

**Fact 2.9** (The Inverse Function Theorem). *Let  $U$  be an open subset of  $\mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^n$  be a  $C^r$ -map. If  $p \in U$  and the derivative*

$$DF_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

*of  $F$  at  $p$  is invertible then there exist open neighbourhoods  $U_p$  around  $p$  and  $U_q$  around  $q = F(p)$  such that  $\hat{F} = F|_{U_p} : U_p \rightarrow U_q$  is bijective and the inverse  $(\hat{F})^{-1} : U_q \rightarrow U_p$  is a  $C^r$ -map. The derivative  $D(\hat{F}^{-1})_q$  of  $\hat{F}^{-1}$  at  $q$  satisfies*

$$D(\hat{F}^{-1})_q = (DF_p)^{-1}$$

*i.e. it is the inverse of the derivative  $DF_p$  of  $F$  at  $p$ .*

Before stating the implicit function theorem we remind the reader of the definition of the following notions.

**Definition 2.10.** Let  $m, n$  be positive natural numbers,  $U$  be an open subset of  $\mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^m$  be a  $C^r$ -map. A point  $p \in U$  is said to be **critical** for  $F$  if the derivative

$$DF_p : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is not of full rank, and **regular** if it is not critical. A point  $q \in F(U)$  is said to be a **regular value** of  $F$  if every point of the pre-image  $\in F^{-1}(\{q\})$  of  $\{q\}$  is regular and a **critical value** otherwise.

Note that if  $n \geq m$  then  $p \in U$  is a regular point of

$$F = (F_1, \dots, F_m) : U \rightarrow \mathbb{R}^m$$

if and only if the gradients  $\nabla F_1, \dots, \nabla F_m$  of the coordinate functions  $F_1, \dots, F_m : U \rightarrow \mathbb{R}$  are linearly independent at  $p$ , or equivalently, the derivative  $DF_p$  of  $F$  at  $p$  satisfies the following condition

$$\det[DF_p \cdot (DF_p)^t] \neq 0.$$

**Theorem 2.11** (The Implicit Function Theorem). *Let  $m, n$  be natural numbers such that  $n > m$  and  $F : U \rightarrow \mathbb{R}^m$  be a  $C^r$ -map from an open subset  $U$  of  $\mathbb{R}^n$ . If  $q \in F(U)$  is a regular value of  $F$  then the pre-image  $F^{-1}(\{q\})$  of  $q$  is an  $(n - m)$ -dimensional submanifold of  $\mathbb{R}^n$  of class  $C^r$ .*

PROOF. Let  $p$  be an element of  $F^{-1}(\{q\})$  and  $K_p$  be the kernel of the derivative  $DF_p$  i.e. the  $(n-m)$ -dimensional subspace of  $\mathbb{R}^n$  given by  $K_p = \{v \in \mathbb{R}^n \mid DF_p \cdot v = 0\}$ . Let  $\pi_p : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$  be a linear map such that  $\pi_p|_{K_p} : K_p \rightarrow \mathbb{R}^{n-m}$  is bijective and define  $G_p : U \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}$  by

$$G_p : x \mapsto (F(x), \pi_p(x)).$$

Then the derivative  $(DG_p)_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$  of  $G_p$ , with respect to the decomposition  $\mathbb{R}^n = K_p^\perp \oplus K_p$ , is given by

$$(DG_p)_p = \begin{pmatrix} DF_p|_{K_p^\perp} & 0 \\ 0 & \pi_p \end{pmatrix},$$

hence bijective. It now follows from the inverse function theorem that there exist open neighbourhoods  $V_p$  around  $p$  and  $W_p$  around  $G_p(p)$  such that  $\hat{G}_p = G_p|_{V_p} : V_p \rightarrow W_p$  is bijective, the inverse  $\hat{G}_p^{-1} : W_p \rightarrow V_p$  is  $C^r$ ,  $D(\hat{G}_p^{-1})_{G_p(p)} = (DG_p)_p^{-1}$  and  $D(\hat{G}_p^{-1})_y$  is bijective for all  $y \in W_p$ . Now put  $\tilde{U}_p = F^{-1}(\{q\}) \cap V_p$  then

$$\tilde{U}_p = \hat{G}_p^{-1}(\{q\} \times \mathbb{R}^{n-m} \cap W_p)$$

so if  $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n-m}$  is the natural projection onto the second factor, then the map

$$\tilde{x}_p = \pi \circ G_p : \tilde{U}_p \rightarrow (\{q\} \times \mathbb{R}^{n-m}) \cap W_p \rightarrow \mathbb{R}^{n-m}$$

is a chart on the open neighbourhood  $\tilde{U}_p$  of  $p$ . The point  $q \in F(U)$  is a regular value so the set

$$\mathcal{B} = \{(\tilde{U}_p, \tilde{x}_p) \mid p \in F^{-1}(\{q\})\}$$

is a  $C^r$ -atlas for  $F^{-1}(\{q\})$ .  $\square$

As a direct consequence of the implicit function theorem we have the following examples of the  $m$ -dimensional sphere  $S^m$  and its  $2m$ -dimensional tangent bundle  $TS^m$  as differentiable submanifolds of  $\mathbb{R}^{m+1}$  and  $\mathbb{R}^{2m+2}$ , respectively.

**Example 2.12.** Let  $F : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$  be the  $C^\omega$ -map given by

$$F : (p_1, \dots, p_{m+1}) \mapsto \sum_{i=1}^{m+1} p_i^2.$$

The derivative  $DF_p$  of  $F$  at  $p$  is given by  $DF_p = 2p$ , so

$$DF_p \cdot (DF_p)^t = 4|p|^2 \in \mathbb{R}.$$

This means that  $1 \in \mathbb{R}$  is a regular value of  $F$  so the fibre

$$S^m = \{p \in \mathbb{R}^{m+1} \mid |p|^2 = 1\} = F^{-1}(\{1\})$$

of  $F$  is an  $m$ -dimensional submanifold of  $\mathbb{R}^{m+1}$ . It is of course the standard  $m$ -dimensional sphere introduced in Example 2.3.

**Example 2.13.** Let  $F : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}^2$  be the  $C^\omega$ -map defined by  $F : (p, v) \mapsto ((|p|^2 - 1)/2, \langle p, v \rangle)$ . The derivative  $DF_{(p,v)}$  of  $F$  at  $(p, v)$  satisfies

$$DF_{(p,v)} = \begin{pmatrix} p & 0 \\ v & p \end{pmatrix}.$$

Hence

$$\det[DF \cdot (DF)^t] = |p|^2(|p|^2 + |v|^2) = 1 + |v|^2 > 0$$

on  $F^{-1}(\{0\})$ . This means that

$$F^{-1}(\{0\}) = \{(p, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \mid |p|^2 = 1 \text{ and } \langle p, v \rangle = 0\}$$

which we denote by  $TS^m$  is a  $2m$ -dimensional submanifold of  $\mathbb{R}^{2m+2}$ . We shall see later that  $TS^m$  is what is called the tangent bundle of the  $m$ -dimensional sphere.

We shall now use the implicit function theorem to construct the important orthogonal group  $\mathbf{O}(m)$  as a submanifold of the set of real  $m \times m$  matrices  $\mathbb{R}^{m \times m}$ .

**Example 2.14.** Let  $\text{Sym}(\mathbb{R}^m)$  be the linear subspace of  $\mathbb{R}^{m \times m}$  consisting of all symmetric  $m \times m$ -matrices

$$\text{Sym}(\mathbb{R}^m) = \{A \in \mathbb{R}^{m \times m} \mid A = A^t\}.$$

Then it is easily seen that the dimension of  $\text{Sym}(\mathbb{R}^m)$  is  $m(m+1)/2$ . Let  $F : \mathbb{R}^{m \times m} \rightarrow \text{Sym}(\mathbb{R}^m)$  be the map defined by  $F : A \mapsto AA^t$ . Then the differential  $DF_A$  of  $F$  at  $A \in \mathbb{R}^{m \times m}$  satisfies

$$DF_A : X \mapsto AX^t + XA^t.$$

This means that for an arbitrary element  $A$  of

$$\mathbf{O}(m) = F^{-1}(\{e\}) = \{A \in \mathbb{R}^{m \times m} \mid AA^t = e\}$$

and  $Y \in \text{Sym}(\mathbb{R}^m)$  we have  $DF_A(YA/2) = Y$ . Hence the differential  $DF_A$  is surjective, so the identity matrix  $e \in \text{Sym}(\mathbb{R}^m)$  is a regular value of  $F$ . Following the implicit function theorem  $\mathbf{O}(m)$  is an  $m(m-1)/2$ -dimensional submanifold of  $\mathbb{R}^{m \times m}$ . The set  $\mathbf{O}(m)$  is the well known **orthogonal group**.

The concept of a differentiable map  $U \rightarrow \mathbb{R}^n$ , defined on an open subset of  $\mathbb{R}^m$ , can be generalized to mappings between manifolds. We shall see that the most important properties of these objects in the classical case are also valid in the manifold setting.

**Definition 2.15.** Let  $(M^m, \hat{\mathcal{A}})$  and  $(N^n, \hat{\mathcal{B}})$  be two  $C^r$ -manifolds. A map  $\phi : M \rightarrow N$  is said to be **differentiable** of class  $C^r$  if for all charts  $(U, x) \in \hat{\mathcal{A}}$  and  $(V, y) \in \hat{\mathcal{B}}$  the map

$$y \circ \phi \circ x^{-1}|_{x(U \cap \phi^{-1}(V))} : x(U \cap \phi^{-1}(V)) \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is of class  $C^r$ . A differentiable map  $\gamma : I \rightarrow M$  defined on an open interval of  $\mathbb{R}$  is called a differentiable **curve** in  $M$ . A differentiable map  $f : M \rightarrow \mathbb{R}$  with values in  $\mathbb{R}$  is called a differentiable **function** on  $M$ . The set of smooth functions defined on  $M$  is denoted by  $C^\infty(M)$ .

It is an easy exercise, using Definition 2.15, to prove the following result concerning the composition of differentiable maps between manifolds.

**Proposition 2.16.** *Let  $(M_1, \hat{\mathcal{A}}_1), (M_2, \hat{\mathcal{A}}_2), (M_3, \hat{\mathcal{A}}_3)$  be  $C^r$ -manifolds and  $\phi : (M_1, \hat{\mathcal{A}}_1) \rightarrow (M_2, \hat{\mathcal{A}}_2)$ ,  $\psi : (M_2, \hat{\mathcal{A}}_2) \rightarrow (M_3, \hat{\mathcal{A}}_3)$  be differentiable maps of class  $C^r$ , then the composition  $\psi \circ \phi : (M_1, \hat{\mathcal{A}}_1) \rightarrow (M_3, \hat{\mathcal{A}}_3)$  is a differentiable map of class  $C^r$ .*

PROOF. See Exercise 2.5. □

**Definition 2.17.** Two manifolds  $(M_1, \hat{\mathcal{A}}_1)$  and  $(M_2, \hat{\mathcal{A}}_2)$  of class  $C^r$  are said to be **diffeomorphic** if there exists a bijective  $C^r$ -map  $\phi : M_1 \rightarrow M_2$ , such that the inverse  $\phi^{-1} : M_2 \rightarrow M_1$  is of class  $C^r$ . In that case the map  $\phi$  is said to be a **diffeomorphism** between  $(M_1, \hat{\mathcal{A}}_1)$  and  $(M_2, \hat{\mathcal{A}}_2)$ .

It can be shown that the 2-dimensional sphere  $S^2$  in  $\mathbb{R}^3$  and the Riemann sphere, introduced earlier, are diffeomorphic, see Exercise 2.7.

**Definition 2.18.** Two  $C^r$ -structures  $\hat{\mathcal{A}}$  and  $\hat{\mathcal{B}}$  on the same topological manifold  $M$  are said to be **different** if the identity map  $\text{id}_M : (M, \hat{\mathcal{A}}) \rightarrow (M, \hat{\mathcal{B}})$  is not a diffeomorphism.

It can be seen that even the real line  $\mathbb{R}$  carries different differentiable structures, see Exercise 2.6.

**Deep Result 2.19.** *Let  $(M_1^m, \hat{\mathcal{A}}_1), (M_2^m, \hat{\mathcal{A}}_2)$  be two differentiable manifolds of class  $C^r$  and of equal dimensions. If  $M$  and  $N$  are homeomorphic as topological spaces and  $m \leq 3$  then  $(M_1, \hat{\mathcal{A}}_1)$  and  $(M_2, \hat{\mathcal{A}}_2)$  are diffeomorphic.*

The following remarkable result was proved by John Milnor in his famous paper: *Differentiable structures on spheres*, Amer. J. Math. **81** (1959), 962-972.



**Deep Result 2.20.** *The 7-dimensional sphere  $S^7$  has exactly 28 different differentiable structures.*

The next very useful proposition generalizes a classical result from the real analysis of several variables.

**Proposition 2.21.** *Let  $(N_1, \hat{\mathcal{B}}_1)$  and  $(N_2, \hat{\mathcal{B}}_2)$  be two differentiable manifolds of class  $C^r$  and  $M_1, M_2$  be submanifolds of  $N_1$  and  $N_2$ , respectively. If  $\phi : N_1 \rightarrow N_2$  is a differentiable map of class  $C^r$  such that  $\phi(M_1)$  is contained in  $M_2$ , then the restriction  $\phi|_{M_1} : M_1 \rightarrow M_2$  is differentiable of class  $C^r$ .*

PROOF. See Exercise 2.8. □

**Example 2.22.** The result of Proposition 2.21 can be used to show that the following maps are all smooth.

- (i)  $\phi_1 : S^2 \subset \mathbb{R}^3 \rightarrow S^3 \subset \mathbb{R}^4, \phi_1 : (x, y, z) \mapsto (x, y, z, 0)$ ,
- (ii)  $\phi_2 : S^3 \subset \mathbb{C}^2 \rightarrow S^2 \subset \mathbb{C} \times \mathbb{R}, \phi_2 : (z_1, z_2) \mapsto (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$ ,
- (iii)  $\phi_3 : \mathbb{R}^1 \rightarrow S^1 \subset \mathbb{C}, \phi_3 : t \mapsto e^{it}$ ,
- (iv)  $\phi_4 : \mathbb{R}^{m+1} - \{0\} \rightarrow S^m, \phi_4 : x \mapsto x/|x|$ ,
- (v)  $\phi_6 : \mathbb{R}^{m+1} - \{0\} \rightarrow \mathbb{R}P^m, \phi_6 : x \mapsto [x]$ .
- (vi)  $\phi_5 : S^m \rightarrow \mathbb{R}P^m, \phi_5 : x \mapsto [x]$ ,

In differential geometry we are especially interested in differentiable manifolds carrying a group structure compatible with their differentiable structure. Such manifolds are named after the famous mathematician Sophus Lie (1842-1899) and will play an important role throughout this work.

**Definition 2.23.** A **Lie group** is a smooth manifold  $G$  with a group structure  $\cdot$  such that the map  $\rho : G \times G \rightarrow G$  with

$$\rho : (p, q) \mapsto p \cdot q^{-1}$$

is smooth. For an element  $p$  of  $G$  the **left translation** by  $p$  is the map  $L_p : G \rightarrow G$  defined by  $L_p : q \mapsto p \cdot q$ .

Note that the standard differentiable  $\mathbb{R}^m$  equipped with the usual addition  $+$  forms an abelian Lie group  $(\mathbb{R}^m, +)$ .

**Corollary 2.24.** *Let  $G$  be a Lie group and  $p$  be an element of  $G$ . Then the left translation  $L_p : G \rightarrow G$  is a smooth diffeomorphism.*

PROOF. See Exercise 2.10. □

**Proposition 2.25.** *Let  $(G, \cdot)$  be a Lie group and  $K$  be a submanifold of  $G$  which is a subgroup. Then  $(K, \cdot)$  is a Lie group.*

**PROOF.** The statement is a direct consequence of Definition 2.23 and Proposition 2.21.  $\square$

The set of non-zero complex numbers  $\mathbb{C}^*$  together with the standard multiplication  $\cdot$  forms a Lie group  $(\mathbb{C}^*, \cdot)$ . The unit circle  $(S^1, \cdot)$  is an interesting compact Lie subgroup of  $(\mathbb{C}^*, \cdot)$ . Another subgroup is the set of the non-zero real numbers  $(\mathbb{R}^*, \cdot)$  containing the positive real numbers  $(\mathbb{R}^+, \cdot)$  and the 0-dimensional sphere  $(S^0, \cdot)$  as subgroups.

**Example 2.26.** Let  $\mathbb{H}$  be the set of quaternions defined by

$$\mathbb{H} = \{z + wj \mid z, w \in \mathbb{C}\}$$

equipped with the conjugation  $\bar{\cdot}$ , addition  $+$  and multiplication  $\cdot$

- (i)  $\overline{(z + wj)} = \bar{z} - wj$ ,
- (ii)  $(z_1 + w_1j) + (z_2 + w_2j) = (z_1 + z_2) + (w_1 + w_2)j$ ,
- (iii)  $(z_1 + w_1j) \cdot (z_2 + w_2j) = (z_1z_2 - w_1\bar{w}_2) + (z_1w_2 + w_1\bar{z}_2)j$

extending the standard operations on  $\mathbb{R}$  and  $\mathbb{C}$  as subsets of  $\mathbb{H}$ . Then it is easily seen that the non-zero quaternions  $(\mathbb{H}^*, \cdot)$  equipped with the multiplication  $\cdot$  form a Lie group. On  $\mathbb{H}$  we define a scalar product

$$\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}, \quad (p, q) \mapsto p \cdot \bar{q}$$

and a real valued norm given by  $|p|^2 = p \cdot \bar{p}$ . Then the 3-dimensional unit sphere  $S^3$  in  $\mathbb{H} \cong \mathbb{R}^4$  with the restricted multiplication forms a compact Lie subgroup  $(S^3, \cdot)$  of  $(\mathbb{H}^*, \cdot)$ . They are both non-abelian.

We shall now introduce some of the classical real and complex matrix Lie groups. As a reference on this topic we recommend the wonderful book: A. W. Knap, *Lie Groups Beyond an Introduction*, Birkhäuser (2002).

**Example 2.27.** The set of invertible real  $m \times m$  matrices

$$\mathbf{GL}(\mathbb{R}^m) = \{A \in \mathbb{R}^{m \times m} \mid \det A \neq 0\}$$

equipped with the standard matrix multiplication has the structure of a Lie group. It is called the real general linear group and its neutral element  $e$  is the identity matrix. The subset  $\mathbf{GL}(\mathbb{R}^m)$  of  $\mathbb{R}^{m \times m}$  is open so  $\dim \mathbf{GL}(\mathbb{R}^m) = m^2$ .

As a subgroup of  $\mathbf{GL}(\mathbb{R}^m)$  we have the real special linear group  $\mathbf{SL}(\mathbb{R}^m)$  given by

$$\mathbf{SL}(\mathbb{R}^m) = \{A \in \mathbb{R}^{m \times m} \mid \det A = 1\}.$$

We will show in Example 3.13 that the dimension of the submanifold  $\mathbf{SL}(\mathbb{R}^m)$  of  $\mathbb{R}^{m \times m}$  is  $m^2 - 1$ .

Another subgroup of  $\mathbf{GL}(\mathbb{R}^m)$  is the orthogonal group

$$\mathbf{O}(m) = \{A \in \mathbb{R}^{m \times m} \mid A^t \cdot A = e\}.$$

As we have already seen in Example 2.14 the dimension of  $\mathbf{O}(m)$  is  $m(m-1)/2$ .

As a subgroup of  $\mathbf{O}(m)$  and  $\mathbf{SL}(\mathbb{R}^m)$  we have the special orthogonal group  $\mathbf{SO}(m)$  which is defined as

$$\mathbf{SO}(m) = \mathbf{O}(m) \cap \mathbf{SL}(\mathbb{R}^m).$$

It can be shown that  $\mathbf{O}(m)$  is diffeomorphic to  $\mathbf{SO}(m) \times \mathbf{O}(1)$ , see Exercise 2.9. Note that  $\mathbf{O}(1) = \{\pm 1\}$  so  $\mathbf{O}(m)$  can be seen as two copies of  $\mathbf{SO}(m)$ . This means that

$$\dim \mathbf{SO}(m) = \dim \mathbf{O}(m) = m(m-1)/2.$$

**Example 2.28.** The set of invertible complex  $m \times m$  matrices

$$\mathbf{GL}(\mathbb{C}^m) = \{A \in \mathbb{C}^{m \times m} \mid \det A \neq 0\}$$

equipped with the standard matrix multiplication has the structure of a Lie group. It is called the complex general linear group and its neutral element  $e$  is the identity matrix. The subset  $\mathbf{GL}(\mathbb{C}^m)$  of  $\mathbb{C}^{m \times m}$  is open so  $\dim(\mathbf{GL}(\mathbb{C}^m)) = 2m^2$ .

As a subgroup of  $\mathbf{GL}(\mathbb{C}^m)$  we have the complex special linear group  $\mathbf{SL}(\mathbb{C}^m)$  given by

$$\mathbf{SL}(\mathbb{C}^m) = \{A \in \mathbb{C}^{m \times m} \mid \det A = 1\}.$$

The dimension of the submanifold  $\mathbf{SL}(\mathbb{C}^m)$  of  $\mathbb{C}^{m \times m}$  is  $2(m^2 - 1)$ .

Another subgroup of  $\mathbf{GL}(\mathbb{C}^m)$  is the unitary group  $\mathbf{U}(m)$  given by

$$\mathbf{U}(m) = \{A \in \mathbb{C}^{m \times m} \mid \bar{A}^t \cdot A = e\}.$$

Calculations similar to those for the orthogonal group show that the dimension of  $\mathbf{U}(m)$  is  $m^2$ .

As a subgroup of  $\mathbf{U}(m)$  and  $\mathbf{SL}(\mathbb{C}^m)$  we have the special unitary group  $\mathbf{SU}(m)$  which is defined as

$$\mathbf{SU}(m) = \mathbf{U}(m) \cap \mathbf{SL}(\mathbb{C}^m).$$

It can be shown that  $\mathbf{U}(1)$  is diffeomorphic to the circle  $S^1$  and that  $\mathbf{U}(m)$  is diffeomorphic to  $\mathbf{SU}(m) \times \mathbf{U}(1)$ , see Exercise 2.9. This means that  $\dim \mathbf{SU}(m) = m^2 - 1$ .

For the rest of this manuscript we shall assume, when not stating otherwise, that our manifolds and maps are smooth i.e. in the  $C^\infty$ -category.

## Exercises

**Exercise 2.1.** Find a proof for Proposition 2.6.

**Exercise 2.2.** Find a proof for Proposition 2.8.

**Exercise 2.3.** Let  $S^1$  be the unit circle in the complex plane  $\mathbb{C}$  given by  $S^1 = \{z \in \mathbb{C} \mid |z|^2 = 1\}$ . Use the maps  $x : \mathbb{C} - \{1\} \rightarrow \mathbb{C}$  and  $y : \mathbb{C} - \{-1\} \rightarrow \mathbb{C}$  with

$$x : z \mapsto \frac{1+z}{i(1-z)}, \quad y : z \mapsto \frac{i(1-z)}{z+1}$$

to show that  $S^1$  is a 1-dimensional submanifold of  $\mathbb{C} \cong \mathbb{R}^2$ .

**Exercise 2.4.** Use the implicit function theorem to show that the  $m$ -dimensional **torus**

$$T^m = \{z \in \mathbb{C}^m \mid |z_1| = \cdots = |z_m| = 1\}$$

is a differentiable submanifold of  $\mathbb{C}^m \cong \mathbb{R}^{2m}$ .

**Exercise 2.5.** Find a proof of Proposition 2.16.

**Exercise 2.6.** Let the real line  $\mathbb{R}$  be equipped with the standard topology and the two  $C^\omega$ -structures  $\hat{\mathcal{A}}$  and  $\hat{\mathcal{B}}$  defined by the following atlases

$$\mathcal{A} = \{(\mathbb{R}, \text{id}_{\mathbb{R}}) \mid \text{id}_{\mathbb{R}} : x \mapsto x\} \quad \text{and} \quad \mathcal{B} = \{(\mathbb{R}, \psi) \mid \psi : x \mapsto x^3\}.$$

Prove that the differentiable structures  $\hat{\mathcal{A}}$  and  $\hat{\mathcal{B}}$  are different but that the differentiable manifolds  $(\mathbb{R}, \hat{\mathcal{A}})$  and  $(\mathbb{R}, \hat{\mathcal{B}})$  are diffeomorphic.

**Exercise 2.7.** Prove that the 2-dimensional sphere  $S^2$  as a differentiable submanifold of the standard  $\mathbb{R}^3$  and the Riemann sphere  $\hat{\mathbb{C}}$  are diffeomorphic.

**Exercise 2.8.** Find a proof of Proposition 2.21.

**Exercise 2.9.** Let the spheres  $S^1$ ,  $S^3$  and the Lie groups  $\mathbf{SO}(n)$ ,  $\mathbf{O}(n)$ ,  $\mathbf{SU}(n)$ ,  $\mathbf{U}(n)$  be equipped with their standard differentiable structures introduced above. Use Proposition 2.21 to prove the following diffeomorphisms

$$S^1 \cong \mathbf{SO}(2), \quad S^3 \cong \mathbf{SU}(2),$$

$$\mathbf{SO}(n) \times \mathbf{O}(1) \cong \mathbf{O}(n), \quad \mathbf{SU}(n) \times \mathbf{U}(1) \cong \mathbf{U}(n).$$

**Exercise 2.10.** Find a proof of Corollary 2.24.

**Exercise 2.11.** Let  $(G, *)$  and  $(H, \cdot)$  be two Lie groups. Prove that the product manifold  $G \times H$  has the structure of a Lie group.

## The Tangent Space

In this chapter we show how a tangent vector at a point  $p \in \mathbb{R}^m$  can be interpreted as a first order linear differential operator, annihilating constants, acting on real valued functions locally defined around  $p$ . This idea is then used to define the notion of the tangent space  $T_p M$  of a manifold  $M$  at a point  $p$  of  $M$ . It turns out that this is a vector space isomorphic to  $\mathbb{R}^m$  where  $m$  is the dimension of  $M$ .

Let  $\mathbb{R}^m$  be the  $m$ -dimensional real vector space with the standard differentiable structure. If  $p$  is a point in  $\mathbb{R}^m$  and  $\gamma : I \rightarrow \mathbb{R}^m$  is a  $C^1$ -curve such that  $\gamma(0) = p$  then the **tangent vector**

$$\dot{\gamma}(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$$

of  $\gamma$  at 0 is an element of  $\mathbb{R}^m$ . Conversely, for an arbitrary element  $v$  of  $\mathbb{R}^m$  we can easily find curves  $\gamma : I \rightarrow \mathbb{R}^m$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . One example is given by

$$\gamma : t \mapsto p + t \cdot v.$$

This shows that the **tangent space**, i.e. the space of tangent vectors, at the point  $p \in \mathbb{R}^m$  can be identified with  $\mathbb{R}^m$ .

We shall now describe how first order differential operators annihilating constants can be interpreted as tangent vectors. For a point  $p$  in  $\mathbb{R}^m$  we denote by  $\varepsilon(p)$  the set of differentiable real-valued functions defined locally around  $p$ . Then it is well known from multi-variable analysis that if  $v \in \mathbb{R}^m$  and  $f \in \varepsilon(p)$  then the **directional derivative**  $\partial_v f$  of  $f$  at  $p$  in the direction of  $v$  is given by

$$\partial_v f = \lim_{t \rightarrow 0} \frac{f(p + tv) - f(p)}{t}.$$

Furthermore the directional derivative  $\partial$  has the following properties:

$$\begin{aligned} \partial_v(\lambda \cdot f + \mu \cdot g) &= \lambda \cdot \partial_v f + \mu \cdot \partial_v g, \\ \partial_v(f \cdot g) &= \partial_v f \cdot g(p) + f(p) \cdot \partial_v g, \\ \partial_{(\lambda \cdot v + \mu \cdot w)} f &= \lambda \cdot \partial_v f + \mu \cdot \partial_w f \end{aligned}$$

for all  $\lambda, \mu \in \mathbb{R}$ ,  $v, w \in \mathbb{R}^m$  and  $f, g \in \varepsilon(p)$ .

**Definition 3.1.** For a point  $p$  in  $\mathbb{R}^m$  let  $T_p\mathbb{R}^m$  be the set of first order linear differential operators at  $p$  annihilating constants i.e. the set of mappings  $\alpha : \varepsilon(p) \rightarrow \mathbb{R}$  such that

- (i)  $\alpha(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \alpha(f) + \mu \cdot \alpha(g)$ ,
- (ii)  $\alpha(f \cdot g) = \alpha(f) \cdot g(p) + f(p) \cdot \alpha(g)$

for all  $\lambda, \mu \in \mathbb{R}$  and  $f, g \in \varepsilon(p)$ .

The set  $T_p\mathbb{R}^m$  carries the natural operations  $+$  and  $\cdot$  transforming it into a real vector space i.e. for all  $\alpha, \beta \in T_pM$ ,  $f \in \varepsilon(p)$  and  $\lambda \in \mathbb{R}$  we have

$$\begin{aligned}(\alpha + \beta)(f) &= \alpha(f) + \beta(f), \\ (\lambda \cdot \alpha)(f) &= \lambda \cdot \alpha(f).\end{aligned}$$

The above mentioned properties of the directional derivative  $\partial$  show that we have a well defined linear map  $\Phi : \mathbb{R}^m \rightarrow T_p\mathbb{R}^m$  given by

$$\Phi : v \mapsto \partial_v.$$

The next result shows that the tangent space at  $p$  can be identified with  $T_p\mathbb{R}^m$  i.e. the space of first order linear operators annihilating constants.

**Theorem 3.2.** For a point  $p$  in  $\mathbb{R}^m$  the map  $\Phi : \mathbb{R}^m \rightarrow T_p\mathbb{R}^m$  defined by  $\Phi : v \mapsto \partial_v$  is a vector space isomorphism.

**PROOF.** Let  $v, w \in \mathbb{R}^m$  such that  $v \neq w$ . Choose an element  $u \in \mathbb{R}^m$  such that  $\langle u, v \rangle \neq \langle u, w \rangle$  and define  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  by  $f(x) = \langle u, x \rangle$ . Then  $\partial_v f = \langle u, v \rangle \neq \langle u, w \rangle = \partial_w f$  so  $\partial_v \neq \partial_w$ . This proves that the map  $\Phi$  is injective.

Let  $\alpha$  be an arbitrary element of  $T_p\mathbb{R}^m$ . For  $k = 1, \dots, m$  let  $\hat{x}_k : \mathbb{R}^m \rightarrow \mathbb{R}$  be the map given by

$$\hat{x}_k : (x_1, \dots, x_m) \mapsto x_k$$

and put  $v_k = \alpha(\hat{x}_k)$ . For the constant function  $1 : (x_1, \dots, x_m) \mapsto 1$  we have

$$\alpha(1) = \alpha(1 \cdot 1) = 1 \cdot \alpha(1) + 1 \cdot \alpha(1) = 2 \cdot \alpha(1),$$

so  $\alpha(1) = 0$ . By the linearity of  $\alpha$  it follows that  $\alpha(c) = 0$  for any constant  $c \in \mathbb{R}$ . Let  $f \in \varepsilon(p)$  and following Lemma 3.3 locally write

$$f(x) = f(p) + \sum_{k=1}^m (\hat{x}_k(x) - p_k) \cdot \psi_k(x),$$

where  $\psi_k \in \varepsilon(p)$  with

$$\psi_k(p) = \frac{\partial f}{\partial x_k}(p).$$

We can now apply the differential operator  $\alpha \in T_p\mathbb{R}^m$  and yield

$$\begin{aligned}\alpha(f) &= \alpha\left(f(p) + \sum_{k=1}^m (\hat{x}_k - p_k) \cdot \psi_k\right) \\ &= \alpha(f(p)) + \sum_{k=1}^m \alpha(\hat{x}_k - p_k) \cdot \psi_k(p) + \sum_{k=1}^m (\hat{x}_k(p) - p_k) \cdot \alpha(\psi_k) \\ &= \sum_{k=1}^m v_k \frac{\partial f}{\partial x_k}(p) \\ &= \partial_v f,\end{aligned}$$

where  $v = (v_1, \dots, v_m) \in \mathbb{R}^m$ . This means that  $\Phi(v) = \partial_v = \alpha$  so the map  $\Phi : \mathbb{R}^m \rightarrow T_p\mathbb{R}^m$  is surjective and hence a vector space isomorphism.  $\square$

**Lemma 3.3.** *Let  $p$  be a point in  $\mathbb{R}^m$  and  $f : U \rightarrow \mathbb{R}$  be a function defined on an open ball around  $p$ . Then for each  $k = 1, 2, \dots, m$  there exist functions  $\psi_k : U \rightarrow \mathbb{R}$  such that*

$$f(x) = f(p) + \sum_{k=1}^m (x_k - p_k) \cdot \psi_k(x) \quad \text{and} \quad \psi_k(p) = \frac{\partial f}{\partial x_k}(p)$$

for all  $x \in U$ .

PROOF. It follows from the fundamental theorem of calculus that

$$\begin{aligned}f(x) - f(p) &= \int_0^1 \frac{\partial}{\partial t} (f(p + t(x - p))) dt \\ &= \sum_{k=1}^m (x_k - p_k) \cdot \int_0^1 \frac{\partial f}{\partial x_k} (p + t(x - p)) dt.\end{aligned}$$

The statement then immediately follows by setting

$$\psi_k(x) = \int_0^1 \frac{\partial f}{\partial x_k} (p + t(x - p)) dt.$$

$\square$

Let  $p$  be a point in  $\mathbb{R}^m$ ,  $v \in \mathbb{R}^m$  be a tangent vector at  $p$  and  $f : U \rightarrow \mathbb{R}$  be a  $C^1$ -function defined on an open subset  $U$  of  $\mathbb{R}^m$  containing  $p$ . Let  $\gamma : I \rightarrow U$  be a curve such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . The identification given by Theorem 3.2 tells us that  $v$  acts on  $f$  by

$$v(f) = \frac{d}{dt} (f \circ \gamma(t))|_{t=0} = df_p(\dot{\gamma}(0)) = \langle \text{grad} f_p, \dot{\gamma}(0) \rangle = \langle \text{grad} f_p, v \rangle.$$

Note that the real number  $v(f)$  is independent of the choice of the curve  $\gamma$  as long as  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . As a direct consequence of Theorem 3.2 we have the following useful result.

**Corollary 3.4.** *Let  $p$  be a point in  $\mathbb{R}^m$  and  $\{e_k \mid k = 1, \dots, m\}$  be a basis for  $\mathbb{R}^m$ . Then the set  $\{\partial_{e_k} \mid k = 1, \dots, m\}$  is a basis for the tangent space  $T_p\mathbb{R}^m$  at  $p$ .*

We shall now use the ideas presented above to generalize to the manifold setting. Let  $M$  be a differentiable manifold and for a point  $p \in M$  let  $\varepsilon(p)$  denote the set of differentiable functions defined on an open neighborhood of  $p$ .

**Definition 3.5.** Let  $M$  be a differentiable manifold and  $p$  be a point on  $M$ . A **tangent vector**  $X_p$  at  $p$  is a map  $X_p : \varepsilon(p) \rightarrow \mathbb{R}$  such that

- (i)  $X_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot X_p(f) + \mu \cdot X_p(g)$ ,
- (ii)  $X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)$

for all  $\lambda, \mu \in \mathbb{R}$  and  $f, g \in \varepsilon(p)$ . The set of tangent vectors at  $p$  is called the **tangent space** at  $p$  and denoted by  $T_pM$ .

The tangent space  $T_pM$  carries the natural operations  $+$  and  $\cdot$  turning it into a real vector space i.e. for all  $X_p, Y_p \in T_pM$ ,  $f \in \varepsilon(p)$  and  $\lambda \in \mathbb{R}$  we have

$$\begin{aligned} (X_p + Y_p)(f) &= X_p(f) + Y_p(f), \\ (\lambda \cdot X_p)(f) &= \lambda \cdot X_p(f). \end{aligned}$$

**Definition 3.6.** Let  $\phi : M \rightarrow N$  be a differentiable map between manifolds. Then the **differential**  $d\phi_p$  of  $\phi$  at a point  $p$  in  $M$  is the map  $d\phi_p : T_pM \rightarrow T_{\phi(p)}N$  such that for all  $X_p \in T_pM$  and  $f \in \varepsilon(\phi(p))$

$$(d\phi_p(X_p))(f) = X_p(f \circ \phi).$$

We shall now use charts to give some motivations for the above definitions and hopefully convince the reader that they are not only abstract nonsense.

**Proposition 3.7.** *Let  $\phi : M \rightarrow N$  and  $\psi : N \rightarrow \bar{N}$  be differentiable maps between manifolds, then for each  $p \in M$  we have*

- (i) *the map  $d\phi_p : T_pM \rightarrow T_{\phi(p)}N$  is linear,*
- (ii) *if  $id_M : M \rightarrow M$  is the identity map, then  $d(id_M)_p = id_{T_pM}$ ,*
- (iii)  *$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$ .*

**PROOF.** The only non-trivial statement is the relation (iii) which is called the **chain rule**. If  $X_p \in T_pM$  and  $f \in \varepsilon(\psi \circ \phi(p))$ , then

$$(d\psi_{\phi(p)} \circ d\phi_p)(X_p)(f) = (d\psi_{\phi(p)}(d\phi_p(X_p)))(f)$$



$$\begin{aligned}
&= (d\phi_p(X_p))(f \circ \psi) \\
&= X_p(f \circ \psi \circ \phi) \\
&= d(\psi \circ \phi)_p(X_p)(f).
\end{aligned}$$

□

**Corollary 3.8.** *Let  $\phi : M \rightarrow N$  be a diffeomorphism with inverse  $\psi = \phi^{-1} : N \rightarrow M$ . Then the differential  $d\phi_p : T_pM \rightarrow T_{\phi(p)}N$  at  $p$  is bijective and  $(d\phi_p)^{-1} = d\psi_{\phi(p)}$ .*

PROOF. The statement is a direct consequence of the following relations

$$\begin{aligned}
d\psi_{\phi(p)} \circ d\phi_p &= d(\psi \circ \phi)_p = d(\text{id}_M)_p = \text{id}_{T_pM}, \\
d\phi_p \circ d\psi_{\phi(p)} &= d(\phi \circ \psi)_{\phi(p)} = d(\text{id}_N)_{\phi(p)} = \text{id}_{T_{\phi(p)}N}.
\end{aligned}$$

□

We are now ready to prove the following interesting result. This is of course a direct generalization of the corresponding result in the classical theory for surfaces in  $\mathbb{R}^3$ .

**Theorem 3.9.** *Let  $M^m$  be an  $m$ -dimensional differentiable manifold and  $p$  be a point on  $M$ . Then the tangent space  $T_pM$  at  $p$  is an  $m$ -dimensional real vector space.*

PROOF. Let  $(U, x)$  be a chart on  $M$ . Then the linear map  $dx_p : T_pM \rightarrow T_{x(p)}\mathbb{R}^m$  is a vector space isomorphism. The statement now follows from Theorem 3.2 and Corollary 3.8. □

Let  $p$  be a point on an  $m$ -dimensional manifold  $M$ ,  $X_p$  be an element of the tangent space  $T_pM$  and  $(U, x)$  be a chart around  $p$ . The differential  $dx_p : T_pM \rightarrow T_{x(p)}\mathbb{R}^m$  is a bijective linear map so there exists a tangent vector  $v$  in  $\mathbb{R}^m$  such that  $dx_p(X_p) = v$ . The image  $x(U)$  is an open subset of  $\mathbb{R}^m$  containing  $x(p)$  so we can find a curve  $c : (-\epsilon, \epsilon) \rightarrow x(U)$  with  $c(0) = x(p)$  and  $\dot{c}(0) = v$ . Then the composition  $\gamma = x^{-1} \circ c : (-\epsilon, \epsilon) \rightarrow U$  is a curve in  $M$  through  $p$  since  $\gamma(0) = p$ . The element  $d(x^{-1})_{x(p)}(v)$  of the tangent space  $T_pM$  denoted by  $\dot{\gamma}(0)$  is called the **tangent to the curve**  $\gamma$  at  $p$ . It follows from the relation

$$\dot{\gamma}(0) = d(x^{-1})_{x(p)}(v) = X_p$$

that the tangent space  $T_pM$  can be thought of as the set of tangents to curves through the point  $p$ .

If  $f : U \rightarrow \mathbb{R}$  is a  $C^1$ -function defined locally on  $U$  then it follows from Definition 3.6 that

$$X_p(f) = (dx_p(X_p))(f \circ x^{-1})$$

$$\begin{aligned}
&= \frac{d}{dt}(f \circ x^{-1} \circ c(t))|_{t=0} \\
&= \frac{d}{dt}(f \circ \gamma(t))|_{t=0}
\end{aligned}$$

It should be noted that the value  $X_p(f)$  is independent of the choice of the chart  $(U, x)$  around  $p$  and the curve  $c : I \rightarrow x(U)$  as long as  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X_p$ . This leads to the following construction of a basis for tangent spaces.

**Proposition 3.10.** *Let  $M^m$  be a differentiable manifold,  $(U, x)$  be local coordinates on  $M$  and  $\{e_k \mid k = 1, \dots, m\}$  be the canonical basis for  $\mathbb{R}^m$ . For an arbitrary point  $p$  in  $U$  we define  $(\frac{\partial}{\partial x_k})_p$  in  $T_p M$  by*

$$\left(\frac{\partial}{\partial x_k}\right)_p : f \mapsto \frac{\partial f}{\partial x_k}(p) = \partial_{e_k}(f \circ x^{-1})(x(p)).$$

Then the set

$$\left\{\left(\frac{\partial}{\partial x_k}\right)_p \mid k = 1, 2, \dots, m\right\}$$

is a basis for the tangent space  $T_p M$  of  $M$  at  $p$ .

**PROOF.** The local chart  $x : U \rightarrow x(U)$  is a diffeomorphism and the differential  $(dx^{-1})_{x(p)} : T_{x(p)}\mathbb{R}^m \rightarrow T_p M$  of its inverse  $x^{-1} : x(U) \rightarrow U$  satisfies

$$\begin{aligned}
(dx^{-1})_{x(p)}(\partial_{e_k})(f) &= \partial_{e_k}(f \circ x^{-1})(x(p)) \\
&= \left(\frac{\partial}{\partial x_k}\right)_p(f)
\end{aligned}$$

for all  $f \in \varepsilon(p)$ . The statement is then a direct consequence of Corollary 3.4.  $\square$

We can now determine the tangent spaces to some of the explicit differentiable manifolds introduced in Chapter 2. We start with the sphere.

**Example 3.11.** Let  $\gamma : (-\epsilon, \epsilon) \rightarrow S^m$  be a curve into the  $m$ -dimensional unit sphere in  $\mathbb{R}^{m+1}$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . The curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \dot{\gamma}(t), \gamma(t) \rangle + \langle \gamma(t), \dot{\gamma}(t) \rangle = 0.$$

This means that  $\langle v, p \rangle = 0$  so every tangent vector  $v \in T_p S^m$  must be orthogonal to  $p$ . On the other hand if  $v \neq 0$  satisfies  $\langle v, p \rangle = 0$  then

$\gamma : \mathbb{R} \rightarrow S^m$  with

$$\gamma : t \mapsto \cos(t|v|) \cdot p + \sin(t|v|) \cdot v/|v|$$

is a curve into  $S^m$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . This shows that the tangent space  $T_p S^m$  is given by

$$T_p S^m = \{v \in \mathbb{R}^{m+1} \mid \langle p, v \rangle = 0\}.$$

In order to determine the tangent spaces of the classical Lie groups we need the differentiable exponential map  $\text{Exp} : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{m \times m}$  for matrices given by the following converging power series

$$\text{Exp} : X \mapsto \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

For this map we have the following well-known result.

**Proposition 3.12.** *Let  $\text{Exp} : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{m \times m}$  be the exponential map for matrices. If  $X, Y \in \mathbb{C}^{m \times m}$ , then*

- (i)  $\det(\text{Exp}(X)) = e^{\text{trace}(X)}$ ,
- (ii)  $\text{Exp}(\bar{X}^t) = \overline{\text{Exp}(X)^t}$ , and
- (ii)  $\text{Exp}(X + Y) = \text{Exp}(X) \cdot \text{Exp}(Y)$  whenever  $XY = YX$ .

PROOF. See Exercise 3.2 □

The real general linear group  $\mathbf{GL}(\mathbb{R}^m)$  is an open subset of  $\mathbb{R}^{m \times m}$  so its tangent space  $T_p \mathbf{GL}(\mathbb{R}^m)$  at any point  $p$  is simply  $\mathbb{R}^{m \times m}$ . The tangent space  $T_e \mathbf{SL}(\mathbb{R}^m)$  of the special linear group  $\mathbf{SL}(\mathbb{R}^m)$  at the neutral element  $e$  can be determined as follows.

**Example 3.13.** For a real  $m \times m$  matrix  $X$  with  $\text{trace}(X) = 0$  define a curve  $A : \mathbb{R} \rightarrow \mathbb{R}^{m \times m}$  by

$$A : s \mapsto \text{Exp}(s \cdot X).$$

Then  $A(0) = e$ ,  $A'(0) = X$  and

$$\det(A(s)) = \det(\text{Exp}(s \cdot X)) = e^{\text{trace}(s \cdot X)} = e^0 = 1.$$

This shows that  $A$  is a curve into the special linear group  $\mathbf{SL}(\mathbb{R}^m)$  and that  $X$  is an element of the tangent space  $T_e \mathbf{SL}(\mathbb{R}^m)$  of  $\mathbf{SL}(\mathbb{R}^m)$  at the neutral element  $e$ . Hence the linear space

$$\{X \in \mathbb{R}^{m \times m} \mid \text{trace}(X) = 0\}$$

of dimension  $m^2 - 1$  is contained in the tangent space  $T_e \mathbf{SL}(\mathbb{R}^m)$ .

The curve given by  $s \mapsto \text{Exp}(se) = \exp(s)e$  is not contained in  $\mathbf{SL}(\mathbb{R}^m)$  so the dimension of  $T_e \mathbf{SL}(\mathbb{R}^m)$  is at most  $m^2 - 1$ . This shows that

$$T_e \mathbf{SL}(\mathbb{R}^m) = \{X \in \mathbb{R}^{m \times m} \mid \text{trace}(X) = 0\}.$$

**Example 3.14.** Let  $A : (-\epsilon, \epsilon) \rightarrow \mathbf{O}(m)$  be a curve into the orthogonal group  $\mathbf{O}(m)$  such that  $A(0) = e$ . Then  $A(s) \cdot A(s)^t = e$  for all  $s \in (-\epsilon, \epsilon)$  and differentiation yields

$$\{A'(s) \cdot A(s)^t + A(s) \cdot A'(s)^t\}|_{s=0} = 0$$

or equivalently  $A'(0) + A'(0)^t = 0$ . This means that each tangent vector of  $\mathbf{O}(m)$  at  $e$  is a skew-symmetric matrix.

On the other hand, for an arbitrary real skew-symmetric matrix  $X$  define the curve  $B : \mathbb{R} \rightarrow \mathbb{R}^{m \times m}$  by  $B : s \mapsto \text{Exp}(s \cdot X)$ , where  $\text{Exp}$  is the exponential map for matrices defined in Exercise 3.2. Then

$$\begin{aligned} B(s) \cdot B(s)^t &= \text{Exp}(s \cdot X) \cdot \text{Exp}(s \cdot X)^t \\ &= \text{Exp}(s \cdot X) \cdot \text{Exp}(s \cdot X^t) \\ &= \text{Exp}(s(X + X^t)) \\ &= \text{Exp}(0) \\ &= e. \end{aligned}$$

This shows that  $B$  is a curve on the orthogonal group,  $B(0) = e$  and  $B'(0) = X$  so  $X$  is an element of  $T_e \mathbf{O}(m)$ . Hence

$$T_e \mathbf{O}(m) = \{X \in \mathbb{R}^{m \times m} \mid X + X^t = 0\}.$$

The dimension of  $T_e \mathbf{O}(m)$  is therefore  $m(m-1)/2$ . This confirms our calculations of the dimension of  $\mathbf{O}(m)$  in Example 2.14 since we know that  $\dim(\mathbf{O}(m)) = \dim(T_e \mathbf{O}(m))$ . The orthogonal group  $\mathbf{O}(m)$  is diffeomorphic to  $\mathbf{SO}(m) \times \{\pm 1\}$  so  $\dim(\mathbf{SO}(m)) = \dim(\mathbf{O}(m))$  hence

$$T_e \mathbf{SO}(m) = T_e \mathbf{O}(m) = \{X \in \mathbb{R}^{m \times m} \mid X + X^t = 0\}.$$

We have proved the following result.

**Proposition 3.15.** *Let  $e$  be the neutral element of the classical real Lie groups  $\mathbf{GL}(\mathbb{R}^m)$ ,  $\mathbf{SL}(\mathbb{R}^m)$ ,  $\mathbf{O}(m)$ ,  $\mathbf{SO}(m)$ . Then their tangent spaces at  $e$  are given by*

$$\begin{aligned} T_e \mathbf{GL}(\mathbb{R}^m) &= \mathbb{R}^{m \times m} \\ T_e \mathbf{SL}(\mathbb{R}^m) &= \{X \in \mathbb{R}^{m \times m} \mid \text{trace}(X) = 0\} \\ T_e \mathbf{O}(m) &= \{X \in \mathbb{R}^{m \times m} \mid X^t + X = 0\} \\ T_e \mathbf{SO}(m) &= T_e \mathbf{O}(m) \cap T_e \mathbf{SL}(\mathbb{R}^m) = T_e \mathbf{O}(m) \end{aligned}$$

For the classical complex Lie groups similar methods can be used to prove the following.

**Proposition 3.16.** *Let  $e$  be the neutral element of the classical complex Lie groups  $\mathbf{GL}(\mathbb{C}^m)$ ,  $\mathbf{SL}(\mathbb{C}^m)$ ,  $\mathbf{U}(m)$ , and  $\mathbf{SU}(m)$ . Then their tangent spaces at  $e$  are given by*

$$\begin{aligned} T_e \mathbf{GL}(\mathbb{C}^m) &= \mathbb{C}^{m \times m} \\ T_e \mathbf{SL}(\mathbb{C}^m) &= \{Z \in \mathbb{C}^{m \times m} \mid \text{trace}(Z) = 0\} \\ T_e \mathbf{U}(m) &= \{Z \in \mathbb{C}^{m \times m} \mid \bar{Z}^t + Z = 0\} \\ T_e \mathbf{SU}(m) &= T_e \mathbf{U}(m) \cap T_e \mathbf{SL}(\mathbb{C}^m). \end{aligned}$$

PROOF. See Exercise 3.4 □

The rest of this chapter is devoted to the introduction of special types of differentiable maps, the so called immersions, embeddings and submersions. If  $\gamma_M : (-\epsilon, \epsilon) \rightarrow M$  is a curve on  $M$  such that  $\gamma_M(0) = p$  then a differentiable map  $\phi : M \rightarrow N$  takes  $\gamma_M$  to a curve

$$\gamma_N = \phi \circ \gamma_M : (-\epsilon, \epsilon) \rightarrow N$$

on  $N$  with  $\gamma_N(0) = \phi(p)$ . The interpretation of the tangent spaces given above shows that the differential  $d\phi_p : T_p M \rightarrow T_{\phi(p)} N$  of  $\phi$  at  $p$  maps the tangent  $\dot{\gamma}_M(0)$  at  $p$  to the tangent  $\dot{\gamma}_N(0)$  at  $\phi(p)$  i. e.

$$d\phi_p(\dot{\gamma}_M(0)) = \dot{\gamma}_N(0).$$

**Definition 3.17.** A differentiable map  $\phi : M \rightarrow N$  between manifolds is said to be an **immersion** if for each  $p \in M$  the differential  $d\phi_p : T_p M \rightarrow T_{\phi(p)} N$  is injective. An **embedding** is an immersion  $\phi : M \rightarrow N$  which is a homeomorphism onto its image  $\phi(M)$ .

For positive integers  $m, n$  with  $m < n$  we have the inclusion map  $\phi : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{n+1}$  given by  $\phi : (x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_{m+1}, 0, \dots, 0)$ . The differential  $d\phi_x$  at  $x$  is injective since  $d\phi_x(v) = (v, 0)$ . The map  $\phi$  is obviously a homeomorphism onto its image  $\phi(\mathbb{R}^{m+1})$  hence an embedding. It is easily seen that even the restriction  $\phi|_{S^m} : S^m \rightarrow S^n$  of  $\phi$  to the  $m$ -dimensional unit sphere  $S^m$  in  $\mathbb{R}^{m+1}$  is an embedding.

**Definition 3.18.** Let  $M$  be an  $m$ -dimensional differentiable manifold and  $U$  be an open subset of  $\mathbb{R}^m$ . An immersion  $\psi : U \rightarrow M$  is called a local **parametrization** of  $M$ .

If  $M$  is a differentiable manifold and  $(U, x)$  a chart on  $M$  then the inverse  $x^{-1} : x(U) \rightarrow U$  of  $x$  is a parametrization of the subset  $U$  of  $M$ .

**Example 3.19.** Let  $S^1$  be the unit circle in the complex plane  $\mathbb{C}$ . For a non-zero integer  $k \in \mathbb{Z}$  define  $\phi_k : S^1 \rightarrow \mathbb{C}$  by  $\phi_k : z \mapsto z^k$ . For a

point  $w \in S^1$  let  $\gamma_w : \mathbb{R} \rightarrow S^1$  be the curve with  $\gamma_w : t \mapsto we^{it}$ . Then  $\gamma_w(0) = w$  and  $\dot{\gamma}_w(0) = iw$ . For the differential of  $\phi_k$  we have

$$(d\phi_k)_w(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_k \circ \gamma_w(t))|_{t=0} = \frac{d}{dt}(w^k e^{ikt})|_{t=0} = kiw^k.$$

This shows that the differential  $(d\phi_k)_w : T_w S^1 \cong \mathbb{R} \rightarrow T_{w^k} \mathbb{C} \cong \mathbb{R}^2$  is injective, so the map  $\phi_k$  is an immersion. It is easily seen that  $\phi_k$  is an embedding if and only if  $k = \pm 1$ .

**Example 3.20.** Let  $q \in S^3$  be a quaternion of unit length and  $\phi_q : S^1 \rightarrow S^3$  be the map defined by  $\phi_q : z \mapsto qz$ . For  $w \in S^1$  let  $\gamma_w : \mathbb{R} \rightarrow S^1$  be the curve given by  $\gamma_w(t) = we^{it}$ . Then  $\gamma_w(0) = w$ ,  $\dot{\gamma}_w(0) = iw$  and  $\phi_q(\gamma_w(t)) = qwe^{it}$ . By differentiating we yield

$$d\phi_q(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_q(\gamma_w(t)))|_{t=0} = \frac{d}{dt}(qwe^{it})|_{t=0} = qiw.$$

Then  $|d\phi_q(\dot{\gamma}_w(0))| = |qwi| = |q||w| = 1$  implies that the differential  $d\phi_q$  is injective. It is easily checked that the immersion  $\phi_q$  is an embedding.

In the next example we construct an interesting embedding of the real projective space  $\mathbb{R}P^m$  into the vector space  $\text{Sym}(\mathbb{R}^{m+1})$  of the real symmetric  $(m+1) \times (m+1)$  matrices.

**Example 3.21.** Let  $m$  be a positive integer and  $S^m$  be the  $m$ -dimensional unit sphere in  $\mathbb{R}^{m+1}$ . For a point  $p \in S^m$  let

$$L_p = \{(s \cdot p) \in \mathbb{R}^{m+1} \mid s \in \mathbb{R}\}$$

be the line through the origin generated by  $p$  and  $\rho_p : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$  be the reflection about the line  $L_p$ . Then  $\rho_p$  is an element of  $\text{End}(\mathbb{R}^{m+1})$  i.e. the set of linear endomorphisms of  $\mathbb{R}^{m+1}$  which can be identified with  $\mathbb{R}^{(m+1) \times (m+1)}$ . It is easily checked that the reflection about the line  $L_p$  is given by

$$\rho_p : q \mapsto 2\langle q, p \rangle p - q.$$

It then follows from the equations

$$\rho_p(q) = 2\langle q, p \rangle p - q = 2p\langle p, q \rangle - q = (2pp^t - e)q$$

that the matrix in  $\mathbb{R}^{(m+1) \times (m+1)}$  corresponding to  $\rho_p$  is just

$$(2pp^t - e).$$

We shall now show that the map  $\phi : S^m \rightarrow \mathbb{R}^{(m+1) \times (m+1)}$  given by

$$\phi : p \mapsto \rho_p$$

is an immersion. Let  $p$  be an arbitrary point on  $S^m$  and  $\alpha, \beta : I \rightarrow S^m$  be two curves meeting at  $p$ , that is  $\alpha(0) = p = \beta(0)$ , with  $a = \dot{\alpha}(0)$  and  $b = \dot{\beta}(0)$ . For  $\gamma \in \{\alpha, \beta\}$  we have

$$\phi \circ \gamma : t \mapsto (q \mapsto 2\langle q, \gamma(t) \rangle \gamma(t) - q)$$

so

$$\begin{aligned} (d\phi)_p(\dot{\gamma}(0)) &= \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} \\ &= (q \mapsto 2\langle q, \dot{\gamma}(0) \rangle \gamma(0) + 2\langle q, \gamma(0) \rangle \dot{\gamma}(0)). \end{aligned}$$

This means that

$$d\phi_p(a) = (q \mapsto 2\langle q, a \rangle p + 2\langle q, p \rangle a)$$

and

$$d\phi_p(b) = (q \mapsto 2\langle q, b \rangle p + 2\langle q, p \rangle b).$$

If  $a \neq b$  then  $d\phi_p(a) \neq d\phi_p(b)$  so the differential  $d\phi_p$  is injective, hence the map  $\phi : S^m \rightarrow \mathbb{R}^{(m+1) \times (m+1)}$  is an immersion.

If the points  $p, q \in S^m$  are linearly independent, then the lines  $L_p$  and  $L_q$  are different. But these are just the eigenspaces of  $\rho_p$  and  $\rho_q$  with the eigenvalue  $+1$ , respectively. This shows that the linear endomorphisms  $\rho_p, \rho_q$  of  $\mathbb{R}^{m+1}$  are different in this case.

On the other hand, if  $p$  and  $q$  are parallel then  $p = \pm q$  hence  $\rho_p = \rho_q$ . This means that the image  $\phi(S^m)$  can be identified with the quotient space  $S^m / \equiv$  where  $\equiv$  is the equivalence relation defined by

$$x \equiv y \text{ if and only if } x = \pm y.$$

This of course is the real projective space  $\mathbb{R}P^m$  so the map  $\phi$  induces an embedding  $\psi : \mathbb{R}P^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  with

$$\psi : [p] \rightarrow \rho_p.$$

For each  $p \in S^m$  the reflection  $\rho_p : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$  about the line  $L_p$  satisfies

$$\rho_p \cdot \rho_p^t = e.$$

This shows that the image  $\psi(\mathbb{R}P^m) = \phi(S^m)$  is not only contained in the linear space  $\text{Sym}(\mathbb{R}^{m+1})$  but also in the orthogonal group  $\mathbf{O}(m+1)$  which we know is a submanifold of  $\mathbb{R}^{(m+1) \times (m+1)}$ .

The following result was proved by Hassler Whitney in his famous paper, *Differentiable Manifolds*, Ann. of Math. **37** (1936), 645-680.

**Deep Result 3.22.** *For  $1 \leq r \leq \infty$  let  $M$  be an  $m$ -dimensional  $C^r$ -manifold. Then there exists a  $C^r$ -embedding  $\phi : M \rightarrow \mathbb{R}^{2m+1}$  into the  $(2m+1)$ -dimensional real vector space  $\mathbb{R}^{2m+1}$ .*

The classical inverse function theorem generalizes to the manifold setting as follows.

**Theorem 3.23** (The Inverse Function Theorem). *Let  $\phi : M \rightarrow N$  be a differentiable map between manifolds with  $\dim M = \dim N$ . If  $p$  is a point in  $M$  such that the differential  $d\phi_p : T_p M \rightarrow T_{\phi(p)} N$  at  $p$  is bijective then there exist open neighborhoods  $U_p$  around  $p$  and  $U_q$  around  $q = \phi(p)$  such that  $\psi = \phi|_{U_p} : U_p \rightarrow U_q$  is bijective and the inverse  $\psi^{-1} : U_q \rightarrow U_p$  is differentiable.*

PROOF. See Exercise 3.8 □

We shall now use Theorem 3.23 to generalize the classical implicit function theorem to manifolds. For this we need the following definition.

**Definition 3.24.** Let  $m, n$  be positive natural numbers and  $\phi : N^n \rightarrow M^m$  be a differentiable map between manifolds. A point  $p \in N$  is said to be **critical** for  $\phi$  if the differential

$$d\phi_p : T_p N \rightarrow T_{\phi(p)} M$$

is not of full rank, and **regular** if it is not critical. A point  $q \in \phi(N)$  is said to be a **regular value** of  $\phi$  if every point of the pre-image  $\phi^{-1}(\{q\})$  of  $\{q\}$  is regular and a **critical value** otherwise.

**Theorem 3.25** (The Implicit Function Theorem). *Let  $\phi : N^n \rightarrow M^m$  be a differentiable map between manifolds such that  $n > m$ . If  $q \in \phi(N)$  is a regular value, then the pre-image  $\phi^{-1}(\{q\})$  of  $q$  is an  $(n-m)$ -dimensional submanifold of  $N^n$ . The tangent space  $T_p \phi^{-1}(\{q\})$  of  $\phi^{-1}(\{q\})$  at  $p$  is the kernel of the differential  $d\phi_p$  i.e.  $T_p \phi^{-1}(\{q\}) = \text{Ker } d\phi_p$ .*

PROOF. Let  $(V_q, \psi_q)$  be a chart on  $M$  with  $q \in V_q$  and  $\psi_q(q) = 0$ . For a point  $p \in \phi^{-1}(\{q\})$  we choose a chart  $(U_p, \psi_p)$  on  $N$  such that  $p \in U_p$ ,  $\psi_p(p) = 0$  and  $\phi(U_p) \subset V_q$ . The differential of the map

$$\hat{\phi} = \psi_q \circ \phi \circ \psi_p^{-1}|_{\psi_p(U_p)} : \psi_p(U_p) \rightarrow \mathbb{R}^m$$

at the point 0 is given by

$$d\hat{\phi}_0 = (d\psi_q)_q \circ d\phi_p \circ (d\psi_p^{-1})_0 : T_0 \mathbb{R}^n \rightarrow T_0 \mathbb{R}^m.$$

The pairs  $(U_p, \psi_p)$  and  $(V_q, \psi_q)$  are charts so the differentials  $(d\psi_q)_q$  and  $(d\psi_p^{-1})_0$  are bijective. This means that the differential  $d\hat{\phi}_0$  is surjective since  $d\phi_p$  is. It then follows from the implicit function theorem 2.11 that  $\psi_p(\phi^{-1}(\{q\}) \cap U_p)$  is an  $(n-m)$ -dimensional submanifold of  $\psi_p(U_p)$ . Hence  $\phi^{-1}(\{q\}) \cap U_p$  is an  $(n-m)$ -dimensional submanifold of  $U_p$ . This



is true for each point  $p \in \phi^{-1}(\{q\})$  so we have proven that  $\phi^{-1}(\{q\})$  is an  $(n - m)$ -dimensional submanifold of  $N^n$ .

Let  $\gamma : (-\epsilon, \epsilon) \rightarrow \phi^{-1}(\{q\})$  be a curve, such that  $\gamma(0) = p$ . Then

$$(d\phi)_p(\dot{\gamma}(0)) = \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} = \frac{dq}{dt}|_{t=0} = 0.$$

This implies that  $T_p\phi^{-1}(\{q\})$  is contained in and has the same dimension as the kernel of  $d\phi_p$ , so  $T_p\phi^{-1}(\{q\}) = \text{Ker } d\phi_p$ .  $\square$

**Definition 3.26.** For positive integers  $m, n$  with  $n \geq m$  a map  $\phi : N^n \rightarrow M^m$  between two manifolds is called a **submersion** if for each  $p \in n$  the differential  $d\phi_p : T_pN \rightarrow T_{\phi(p)}M$  is surjective.

If  $m, n \in \mathbb{N}$  such that  $m < n$  then we have the projection map  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by  $\pi : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$ . Its differential  $d\pi_x$  at a point  $x$  is surjective since

$$d\pi_x(v_1, \dots, v_n) = (v_1, \dots, v_m).$$

This means that the projection is a submersion. An important submersion between spheres is given by the following.

**Example 3.27.** Let  $S^3$  and  $S^2$  be the unit spheres in  $\mathbb{C}^2$  and  $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ , respectively. The **Hopf map**  $\phi : S^3 \rightarrow S^2$  is given by

$$\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2).$$

The map  $\phi$  and its differential  $d\phi_p : T_pS^3 \rightarrow T_{\phi(p)}S^2$  are surjective for each  $p \in S^3$ . This implies that each point  $q \in S^2$  is a regular value and the fibres of  $\phi$  are 1-dimensional submanifolds of  $S^3$ . They are actually the great circles given by

$$\phi^{-1}(\{(2x\bar{y}, |x|^2 - |y|^2)\}) = \{e^{i\theta}(x, y) \mid \theta \in \mathbb{R}\}.$$

This means that the 3-dimensional sphere  $S^3$  is disjoint union of great circles

$$S^3 = \bigcup_{q \in S^2} \phi^{-1}(\{q\}).$$

## Exercises

**Exercise 3.1.** Let  $p$  be an arbitrary point on the unit sphere  $S^{2n+1}$  of  $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ . Determine the tangent space  $T_p S^{2n+1}$  and show that it contains an  $n$ -dimensional complex subspace of  $\mathbb{C}^{n+1}$ .

**Exercise 3.2.** Find a proof for Proposition 3.12

**Exercise 3.3.** Prove that the matrices

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a basis for the tangent space  $T_e \mathbf{SL}(\mathbb{R}^2)$  of the real special linear group  $\mathbf{SL}(\mathbb{R}^2)$  at  $e$ . For each  $k = 1, 2, 3$  find an explicit formula for the curve  $\gamma_k : \mathbb{R} \rightarrow \mathbf{SL}(\mathbb{R}^2)$  given by

$$\gamma_k : s \mapsto \text{Exp}(s \cdot X_k).$$

**Exercise 3.4.** Find a proof for Proposition 3.16.

**Exercise 3.5.** Prove that the matrices

$$Z_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis for the tangent space  $T_e \mathbf{SU}(2)$  of the special unitary group  $\mathbf{SU}(2)$  at  $e$ . For each  $k = 1, 2, 3$  find an explicit formula for the curve  $\gamma_k : \mathbb{R} \rightarrow \mathbf{SU}(2)$  given by

$$\gamma_k : s \mapsto \text{Exp}(s \cdot Z_k).$$

**Exercise 3.6.** For each  $k \in \mathbb{N}_0$  define  $\phi_k : \mathbb{C} \rightarrow \mathbb{C}$  and  $\psi_k : \mathbb{C}^* \rightarrow \mathbb{C}$  by  $\phi_k, \psi_k : z \mapsto z^k$ . For which  $k \in \mathbb{N}_0$  are  $\phi_k, \psi_k$  immersions, submersions or embeddings.

**Exercise 3.7.** Prove that the map  $\phi : \mathbb{R}^m \rightarrow \mathbb{C}^m$  given by

$$\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$$

is a parametrization of the  $m$ -dimensional torus  $T^m$  in  $\mathbb{C}^m$

**Exercise 3.8.** Find a proof for Theorem 3.23

**Exercise 3.9.** Prove that the Hopf-map  $\phi : S^3 \rightarrow S^2$  with  $\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2)$  is a submersion.

## The Tangent Bundle

The main aim of this chapter is to introduce the tangent bundle  $TM$  of a differentiable manifold  $M^m$ . Intuitively this is the object we get by glueing at each point  $p \in M$  the corresponding tangent space  $T_pM$ . The differentiable structure on  $M$  induces a differentiable structure on  $TM$  making it into a differentiable manifold of dimension  $2m$ . The tangent bundle  $TM$  is the most important example of what is called a vector bundle over  $M$ .

**Definition 4.1.** Let  $E$  and  $M$  be topological manifolds and  $\pi : E \rightarrow M$  be a continuous surjective map. The triple  $(E, M, \pi)$  is called an  $n$ -dimensional **topological vector bundle** over  $M$  if

- (i) for each  $p \in M$  the fibre  $E_p = \pi^{-1}(p)$  is an  $n$ -dimensional vector space,
- (ii) for each  $p \in M$  there exists a **bundle chart**  $(\pi^{-1}(U), \psi)$  consisting of the pre-image  $\pi^{-1}(U)$  of an open neighbourhood  $U$  of  $p$  and a homeomorphism  $\psi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  such that for all  $q \in U$  the map  $\psi_q = \psi|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^n$  is a vector space isomorphism.

The  $n$ -dimensional topological vector bundle  $(E, M, \pi)$  over  $M$  is said to be **trivial** if there exists a global bundle chart  $\psi : E \rightarrow M \times \mathbb{R}^n$ . A continuous map  $\sigma : M \rightarrow E$  is called a **section** of the bundle  $(E, M, \pi)$  if  $\pi \circ \sigma(p) = p$  for each  $p \in M$ .

For each positive integer  $n$  and topological manifold  $M$  we have the  $n$ -dimensional vector bundle  $(M \times \mathbb{R}^n, M, \pi)$  where  $\pi : M \times \mathbb{R}^n \rightarrow M$  is the projection map  $\pi : (x, v) \mapsto x$ . The identity map  $\psi : M \times \mathbb{R}^n \rightarrow M \times \mathbb{R}^n$  is a global bundle chart so the bundle is trivial.

**Definition 4.2.** Let  $(E, M, \pi)$  be an  $n$ -dimensional topological vector bundle over  $M$ . A collection

$$\mathcal{B} = \{(\pi^{-1}(U_\alpha), \psi_\alpha) \mid \alpha \in I\}$$

of bundle charts is called a **bundle atlas** for  $(E, M, \pi)$  if  $M = \cup_\alpha U_\alpha$ . For each pair  $(\alpha, \beta)$  there exists a function  $A_{\alpha, \beta} : U_\alpha \cap U_\beta \rightarrow \mathbf{GL}(\mathbb{R}^n)$

such that the corresponding continuous map

$$\psi_\beta \circ \psi_\alpha^{-1}|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha, \beta}(p))(v)).$$

The elements of  $\{A_{\alpha, \beta} \mid \alpha, \beta \in I\}$  are called the **transition maps** of the bundle atlas  $\mathcal{B}$ .

**Definition 4.3.** Let  $E$  and  $M$  be differentiable manifolds and  $\pi : E \rightarrow M$  be a differentiable map such that  $(E, M, \pi)$  is an  $n$ -dimensional topological vector bundle. A bundle atlas  $\mathcal{B}$  for  $(E, M, \pi)$  is said to be **differentiable** if the corresponding transition maps are differentiable. A **differentiable vector bundle** is a topological vector bundle together with a maximal differentiable bundle atlas. A differentiable section of  $(E, M, \pi)$  is called a **vector field**. By  $C^\infty(E)$  we denote the set of all smooth vector fields of  $(E, M, \pi)$ .

From now on we shall, when not stating otherwise, assume that all our vector bundles are smooth.

**Definition 4.4.** Let  $(E, M, \pi)$  be a vector bundle over a manifold  $M$ . Then we define the operations  $+$  and  $\cdot$  on the set  $C^\infty(E)$  of smooth sections of  $(E, M, \pi)$  by

- (i)  $(v + w)_p = v_p + w_p$ ,
- (ii)  $(f \cdot v)_p = f(p) \cdot v_p$

for all  $v, w \in C^\infty(E)$  and  $f \in C^\infty(M)$ . If  $U$  is an open subset of  $M$  then a set  $\{v_1, \dots, v_n\}$  of vector fields  $v_1, \dots, v_n : U \rightarrow E$  on  $U$  is called a **local frame** for  $E$  if for each  $p \in U$  the set  $\{(v_1)_p, \dots, (v_n)_p\}$  is a basis for the vector space  $E_p$ .

The above defined operations make  $C^\infty(E)$  into a module over  $C^\infty(M)$  and in particular a vector space over the real numbers as the constant functions in  $C^\infty(M)$ .

**Example 4.5** (The Tangent Bundle). Let  $M^m$  be a differentiable manifold with maximal atlas  $\hat{\mathcal{A}}$ , the set  $TM$  be given by

$$TM = \{(p, v) \mid p \in M, v \in T_p M\}$$

and  $\pi : TM \rightarrow M$  be the projection map with  $\pi : (p, v) \mapsto p$ . For a given point  $p \in M$  the fibre  $\pi^{-1}(\{p\})$  of  $\pi$  is the  $m$ -dimensional tangent space  $T_p M$  at  $p$ . The triple  $(TM, M, \pi)$  is called the **tangent bundle** of  $M$ . We shall show how  $(TM, M, \pi)$  can be given the structure of a differentiable vector bundle.

For a chart  $x : U \rightarrow \mathbb{R}^m$  in  $\hat{\mathcal{A}}$  we define  $x^* : \pi^{-1}(U) \rightarrow \mathbb{R}^m \times \mathbb{R}^m$  by

$$x^* : \left( p, \sum_{k=1}^m v_k \left( \frac{\partial}{\partial x_k} \right)_p \right) \mapsto (x(p), (v_1, \dots, v_m)).$$

Note that it is a direct consequence of Proposition 3.10 that the map  $x^*$  is well-defined. The collection

$$\{(x^*)^{-1}(W) \subset TM \mid (U, x) \in \hat{\mathcal{A}} \text{ and } W \subset x(U) \times \mathbb{R}^m \text{ open}\}$$

is a basis for a topology  $\mathcal{T}_{TM}$  on  $TM$  and  $(\pi^{-1}(U), x^*)$  is a chart on the  $2m$ -dimensional topological manifold  $(TM, \mathcal{T}_{TM})$ . If  $(U, x)$  and  $(V, y)$  are two charts in  $\hat{\mathcal{A}}$  such that  $p \in U \cap V$ , then the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \rightarrow \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(a, b) \mapsto (y \circ x^{-1}(a), \sum_{k=1}^m \frac{\partial y_1}{\partial x_k}(x^{-1}(a))b_k, \dots, \sum_{k=1}^m \frac{\partial y_m}{\partial x_k}(x^{-1}(a))b_k).$$

We are assuming that  $y \circ x^{-1}$  is differentiable so it follows that  $(y^*) \circ (x^*)^{-1}$  is also differentiable. This means that

$$\mathcal{A}^* = \{(\pi^{-1}(U), x^*) \mid (U, x) \in \hat{\mathcal{A}}\}$$

is a  $C^r$ -atlas on  $TM$  so  $(TM, \widehat{\mathcal{A}^*})$  is a differentiable manifold. It is trivial that the surjective projection map  $\pi : TM \rightarrow M$  is differentiable.

For each point  $p \in M$  the fibre  $\pi^{-1}(\{p\})$  of  $\pi$  is the tangent space  $T_p M$  of  $M$  at  $p$  hence an  $m$ -dimensional vector space. For a chart  $x : U \rightarrow \mathbb{R}^m$  in  $\hat{\mathcal{A}}$  we define  $\bar{x} : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$  by

$$\bar{x} : \left( p, \sum_{k=1}^m v_k \left( \frac{\partial}{\partial x_k} \right)_p \right) \mapsto (p, (v_1, \dots, v_m)).$$

The restriction  $\bar{x}_p = \bar{x}|_{T_p M} : T_p M \rightarrow \{p\} \times \mathbb{R}^m$  to the tangent space  $T_p M$  is given by

$$\bar{x}_p : \sum_{k=1}^m v_k \left( \frac{\partial}{\partial x_k} \right)_p \mapsto (v_1, \dots, v_m),$$

so it is a vector space isomorphism. This implies that the map  $\bar{x} : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$  is a bundle chart. It is not difficult to see that

$$\mathcal{B} = \{(\pi^{-1}(U), \bar{x}) \mid (U, x) \in \hat{\mathcal{A}}\}$$

is a bundle atlas making  $(TM, M, \pi)$  into an  $m$ -dimensional topological vector bundle. It immediately follows from above that  $(TM, M, \pi)$

together with the maximal bundle atlas  $\hat{\mathcal{B}}$  defined by  $\mathcal{B}$  is a differentiable vector bundle. The set of smooth vector fields  $X : M \rightarrow TM$  is denoted by  $C^\infty(TM)$ .

**Example 4.6.** We have seen earlier that the 3-sphere  $S^3$  in  $\mathbb{H} \cong \mathbb{C}^2$  carries a group structure  $\cdot$  given by

$$(z, w) \cdot (\alpha, \beta) = (z\alpha - w\bar{\beta}, z\beta + w\bar{\alpha}).$$

This makes  $(S^3, \cdot)$  into a Lie group with neutral element  $e = (1, 0)$ . Put  $v_1 = (i, 0)$ ,  $v_2 = (0, 1)$  and  $v_3 = (0, i)$  and for  $k = 1, 2, 3$  define the curves  $\gamma_k : \mathbb{R} \rightarrow S^3$  with

$$\gamma_k : t \mapsto \cos t \cdot (1, 0) + \sin t \cdot v_k.$$

Then  $\gamma_k(0) = e$  and  $\dot{\gamma}_k(0) = v_k$  so  $v_1, v_2, v_3$  are elements of the tangent space  $T_e S^3$ . They are linearly independent so they generate  $T_e S^3$ . The group structure on  $S^3$  can be used to extend vectors in  $T_e S^3$  to vector fields on  $S^3$  as follows. For  $p \in S^3$  let  $L_p : S^3 \rightarrow S^3$  be the left translation on  $S^3$  by  $p$  given by  $L_p : q \mapsto p \cdot q$ . Then define the vector fields  $X_1, X_2, X_3 \in C^\infty(TS^3)$  by

$$(X_k)_p = (dL_p)_e(v_k) = \frac{d}{dt}(L_p(\gamma_k(t)))|_{t=0}.$$

It is left as an exercise for the reader to show that at a point  $p = (z, w) \in S^3$  the values of  $X_k$  at  $p$  is given by

$$\begin{aligned} (X_1)_p &= (z, w) \cdot (i, 0) = (iz, -iw), \\ (X_2)_p &= (z, w) \cdot (0, 1) = (-w, z), \\ (X_3)_p &= (z, w) \cdot (0, i) = (iw, iz). \end{aligned}$$

Our next aim is to introduce the Lie bracket on the set of vector fields  $C^\infty(TM)$  on  $M$ .

**Definition 4.7.** Let  $M$  be a smooth manifold. For two vector fields  $X, Y \in C^\infty(TM)$  we define the **Lie bracket**  $[X, Y]_p : C^\infty(M) \rightarrow \mathbb{R}$  of  $X$  and  $Y$  at  $p \in M$  by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

The next result shows that the Lie bracket  $[X, Y]_p$  actually is an element of the tangent space  $T_p M$ .

**Proposition 4.8.** *Let  $M$  be a smooth manifold,  $X, Y \in C^\infty(TM)$  be vector fields on  $M$ ,  $f, g \in C^\infty(M, \mathbb{R})$  and  $\lambda, \mu \in \mathbb{R}$ . Then*

- (i)  $[X, Y]_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot [X, Y]_p(f) + \mu \cdot [X, Y]_p(g)$ ,
- (ii)  $[X, Y]_p(f \cdot g) = [X, Y]_p(f) \cdot g(p) + f(p) \cdot [X, Y]_p(g)$ .

PROOF.

$$\begin{aligned}
& [X, Y]_p(\lambda f + \mu g) \\
&= X_p(Y(\lambda f + \mu g)) - Y_p(X(\lambda f + \mu g)) \\
&= \lambda X_p(Y(f)) + \mu X_p(Y(g)) - \lambda Y_p(X(f)) - \mu Y_p(X(g)) \\
&= \lambda [X, Y]_p(f) + \mu [X, Y]_p(g).
\end{aligned}$$

$$\begin{aligned}
& [X, Y]_p(f \cdot g) \\
&= X_p(Y(f \cdot g)) - Y_p(X(f \cdot g)) \\
&= X_p(f \cdot Y(g) + g \cdot Y(f)) - Y_p(f \cdot X(g) + g \cdot X(f)) \\
&= X_p(f)Y_p(g) + f(p)X_p(Y(g)) + X_p(g)Y_p(f) + g(p)X_p(Y(f)) \\
&\quad - Y_p(f)X_p(g) - f(p)Y_p(X(g)) - Y_p(g)X_p(f) - g(p)Y_p(X(f)) \\
&= f(p)\{X_p(Y(g)) - Y_p(X(g))\} + g(p)\{X_p(Y(f)) - Y_p(X(f))\} \\
&= f(p)[X, Y]_p(g) + g(p)[X, Y]_p(f).
\end{aligned}$$

□

Proposition 4.8 implies that if  $X, Y$  are vector fields on  $M$  then the map  $[X, Y] : M \rightarrow TM$  given by  $[X, Y] : p \mapsto [X, Y]_p$  is a section of the tangent bundle. In Proposition 4.10 we shall prove that this section is smooth. For this we need the following technical lemma.

**Lemma 4.9.** *Let  $M^m$  be a smooth manifold and  $X : M \rightarrow TM$  be a section of  $TM$ . Then the following conditions are equivalent*

- (i) *the section  $X$  is smooth,*
- (ii) *if  $(U, x)$  is a chart on  $M$  then the functions  $a_1, \dots, a_m : U \rightarrow \mathbb{R}$  given by*

$$\sum_{k=1}^m a_k \frac{\partial}{\partial x_k} = X|_U,$$

*are smooth,*

- (iii) *if  $f : V \rightarrow \mathbb{R}$  defined on an open subset  $V$  of  $M$  is smooth, then the function  $X(f) : V \rightarrow \mathbb{R}$  with  $X(f)(p) = X_p(f)$  is smooth.*

PROOF. (i)  $\Rightarrow$  (ii): The functions  $a_k = \pi_{m+k} \circ x^* \circ X|_U : U \rightarrow \mathbb{R}$  are restrictions of compositions of smooth maps so therefore smooth.

(ii)  $\Rightarrow$  (iii): Let  $(U, x)$  be a chart on  $M$  such that  $U$  is contained in  $V$ . By assumption the map

$$X(f)|_U = \sum_{i=1}^m a_i \frac{\partial f}{\partial x_i}$$

is smooth. This is true for each such chart  $(U, x)$  so the function  $X(f)$  is smooth.

(iii)  $\Rightarrow$  (i): Note that the smoothness of the section  $X$  is equivalent to  $x^* \circ X|_U : U \rightarrow \mathbb{R}^{2m}$  being smooth for all charts  $(U, x)$  on  $M$ . On the other hand, this is equivalent to  $x_k^* = \pi_k \circ x^* \circ X|_U : U \rightarrow \mathbb{R}$  being smooth for all  $k = 1, 2, \dots, 2m$  and all charts  $(U, x)$  on  $M$ . It is trivial that the coordinates  $x_k^* = x_k$  for  $k = 1, \dots, m$  are smooth. But  $x_{m+k}^* = a_k = X(x_k)$  for  $k = 1, \dots, m$  hence also smooth by assumption.  $\square$

**Proposition 4.10.** *Let  $M$  be a manifold and  $X, Y \in C^\infty(TM)$  be vector fields on  $M$ . Then the section  $[X, Y] : M \rightarrow TM$  of the tangent bundle given by  $[X, Y] : p \mapsto [X, Y]_p$  is smooth.*

PROOF. Let  $f : M \rightarrow \mathbb{R}$  be an arbitrary smooth function on  $M$  then  $[X, Y](f) = X(Y(f)) - Y(X(f))$  is smooth so it follows from Lemma 4.9 that the section  $[X, Y]$  is smooth.  $\square$

For later use we prove the following useful result.

**Lemma 4.11.** *Let  $M$  be a smooth manifold and  $[\cdot, \cdot]$  be the Lie bracket on the tangent bundle  $TM$ . Then*

- (i)  $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$ ,
- (ii)  $[f \cdot X, Y] = f \cdot [X, Y] - Y(f) \cdot X$

for all  $X, Y \in C^\infty(TM)$  and  $f \in C^\infty(M)$ ,

PROOF. If  $g \in C^\infty(M)$ , then

$$\begin{aligned} [X, f \cdot Y](g) &= X(f \cdot Y(g)) - f \cdot Y(X(g)) \\ &= X(f) \cdot Y(g) + f \cdot X(Y(g)) - f \cdot Y(X(g)) \\ &= (X(f) \cdot Y + f \cdot [X, Y])(g) \end{aligned}$$

This proves the first statement and the second follows from the skew-symmetry of the Lie bracket.  $\square$

**Definition 4.12.** A real vector space  $(V, +, \cdot)$  equipped with an operation  $[\cdot, \cdot] : V \times V \rightarrow V$  is said to be a real **Lie algebra** if the following relations hold

- (i)  $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$ ,
- (ii)  $[X, Y] = -[Y, X]$ ,
- (iii)  $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

for all  $X, Y, Z \in V$  and  $\lambda, \mu \in \mathbb{R}$ . The equation (iii) is called the **Jacobi identity**.

**Theorem 4.13.** *Let  $M$  be a smooth manifold. The vector space  $C^\infty(TM)$  of smooth vector fields on  $M$  equipped with the Lie bracket  $[\cdot, \cdot] : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$  is a real Lie algebra.*



PROOF. See exercise 4.4.  $\square$

If  $\phi : M \rightarrow N$  is a surjective map between differentiable manifolds then two vector fields  $X \in C^\infty(TM)$ ,  $\bar{X} \in C^\infty(TN)$  are said to be  **$\phi$ -related** if  $d\phi_p(X) = \bar{X}_{\phi(p)}$  for all  $p \in M$ . In that case we write  $\bar{X} = d\phi(X)$ .

**Proposition 4.14.** *Let  $\phi : M \rightarrow N$  be a map between differentiable manifolds,  $X, Y \in C^\infty(TM)$ ,  $\bar{X}, \bar{Y} \in C^\infty(TN)$  such that  $\bar{X} = d\phi(X)$  and  $\bar{Y} = d\phi(Y)$ . Then*

$$[\bar{X}, \bar{Y}] = d\phi([X, Y]).$$

PROOF. Let  $f : N \rightarrow \mathbb{R}$  be a smooth function, then

$$\begin{aligned} [\bar{X}, \bar{Y}](f) &= d\phi(X)(d\phi(Y)(f)) - d\phi(Y)(d\phi(X)(f)) \\ &= X(d\phi(Y)(f) \circ \phi) - Y(d\phi(X)(f) \circ \phi) \\ &= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\ &= [X, Y](f \circ \phi) \\ &= d\phi([X, Y])(f). \end{aligned}$$

$\square$

**Proposition 4.15.** *Let  $\phi : M \rightarrow N$  be a smooth bijective map between differentiable manifolds. If  $X, Y \in C^\infty(TM)$  are vector fields on  $M$ , then*

- (i)  $d\phi(X) \in C^\infty(TN)$ ,
- (ii) the map  $d\phi : C^\infty(TM) \rightarrow C^\infty(TN)$  is a Lie algebra homomorphism i.e.  $[d\phi(X), d\phi(Y)] = d\phi([X, Y])$ .

PROOF. The fact that the map  $\phi$  is bijective implies that  $d\phi(X)$  is a section of the tangent bundle. That  $d\phi(X) \in C^\infty(TN)$  follows directly from the fact that

$$d\phi(X)(f)(\phi(p)) = X(f \circ \phi)(p).$$

The last statement is a direct consequence of Proposition 4.14.  $\square$

**Definition 4.16.** Let  $M$  be a smooth manifold. Two vector fields  $X, Y \in C^\infty(TM)$  are said to **commute** if  $[X, Y] = 0$ .

Let  $(U, x)$  be local coordinates on a manifold  $M$  and let

$$\left\{ \frac{\partial}{\partial x_k} \mid k = 1, 2, \dots, m \right\}$$

be the induced local frame for the tangent bundle. For  $k = 1, 2, \dots, m$  the vector field  $\partial/\partial x_k$  is  $x$ -related to the constant coordinate vector

field  $e_k$  in  $\mathbb{R}^m$ . This implies that

$$dx\left(\left[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right]\right) = [e_k, e_l] = 0.$$

Hence the local frame fields commute.

**Definition 4.17.** Let  $G$  be a Lie group with neutral element  $e$ . For  $p \in G$  let  $L_p : G \rightarrow G$  be the left translation by  $p$  with  $L_p : q \mapsto pq$ . A vector field  $X \in C^\infty(TG)$  is said to be **left invariant** if  $dL_p(X) = X$  for all  $p \in G$ , or equivalently,  $X_{pq} = (dL_p)_q(X_q)$  for all  $p, q \in G$ . The set of all left invariant vector fields on  $G$  is called the **Lie algebra** of  $G$  and is denoted by  $\mathfrak{g}$ .

The Lie algebras of the classical Lie groups introduced earlier are denoted by  $\mathfrak{gl}(\mathbb{R}^m)$ ,  $\mathfrak{sl}(\mathbb{R}^m)$ ,  $\mathfrak{o}(m)$ ,  $\mathfrak{so}(m)$ ,  $\mathfrak{gl}(\mathbb{C}^m)$ ,  $\mathfrak{sl}(\mathbb{C}^m)$ ,  $\mathfrak{u}(m)$  and  $\mathfrak{su}(m)$ , respectively.

**Proposition 4.18.** Let  $G$  be a Lie group and  $\mathfrak{g}$  be the Lie algebra of  $G$ . Then  $\mathfrak{g}$  is a Lie **subalgebra** of  $C^\infty(TG)$  i.e. if  $X, Y \in \mathfrak{g}$  then  $[X, Y] \in \mathfrak{g}$ ,

PROOF. If  $p \in G$  then

$$dL_p([X, Y]) = [dL_p(X), dL_p(Y)] = [X, Y]$$

for all  $X, Y \in \mathfrak{g}$ . This proves that the Lie bracket  $[X, Y]$  of two left invariant vector fields  $X, Y$  is left invariant and thereby that  $\mathfrak{g}$  is a Lie subalgebra of  $C^\infty(TG)$ .  $\square$

Note that if  $X$  is a left invariant vector field on  $G$  then

$$X_p = (dL_p)_e(X_e)$$

so the value  $X_p$  of  $X$  at  $p \in G$  is completely determined by the value  $X_e$  of  $X$  at  $e$ . This means that the map  $*$  :  $T_e G \rightarrow \mathfrak{g}$  given by

$$* : X \mapsto (X^* : p \mapsto (dL_p)_e(X))$$

is a vector space isomorphism and that we can define a Lie bracket  $[,] : T_e G \times T_e G \rightarrow T_e G$  on the tangent space  $T_e G$  by

$$[X, Y] = [X^*, Y^*]_e.$$

**Proposition 4.19.** Let  $G$  be one of the classical Lie groups and  $T_e G$  be the tangent space of  $G$  at the neutral element  $e$ . Then the Lie bracket on  $T_e G$

$$[,] : T_e G \times T_e G \rightarrow T_e G$$

is given by

$$[X_e, Y_e] = X_e \cdot Y_e - Y_e \cdot X_e$$

where  $\cdot$  is the usual matrix multiplication.

PROOF. We shall prove the result for the case when  $G$  is the real general linear group  $\mathbf{GL}(\mathbb{R}^m)$ . For the other real classical Lie groups the result follows from the fact that they are all subgroups of  $\mathbf{GL}(\mathbb{R}^m)$ . The same proof can be used for the complex cases.

Let  $X, Y \in \mathfrak{gl}(\mathbb{R}^m)$  be left invariant vector fields on  $\mathbf{GL}(\mathbb{R}^m)$ ,  $f : U \rightarrow \mathbb{R}$  be a function defined locally around the identity element  $e \in \mathbf{GL}(\mathbb{R}^m)$  and  $p$  be an arbitrary point in  $U$ . Then the derivative  $X_p(f)$  is given by

$$X_p(f) = \frac{d}{ds}(f(p \cdot \text{Exp}(sX_e)))|_{s=0} = df_p(p \cdot X_e) = df_p(X_p).$$

The real general linear group  $\mathbf{GL}(\mathbb{R}^m)$  is an open subset of  $\mathbb{R}^{m \times m}$  so we can use well-known rules from calculus and the second derivative  $Y_e(X(f))$  is obtained as follows:

$$\begin{aligned} Y_e(X(f)) &= \frac{d}{dt}(X_{\text{Exp}(tY_e)}(f))|_{t=0} \\ &= \frac{d}{dt}(df_{\text{Exp}(tY_e)}(\text{Exp}(tY_e) \cdot X_e))|_{t=0} \\ &= d^2f_e(Y_e, X_e) + df_e(Y_e \cdot X_e). \end{aligned}$$

The Hessian  $d^2f_e$  of  $f$  is symmetric, hence

$$[X, Y]_e(f) = X_e(Y(f)) - Y_e(X(f)) = df_e(X_e \cdot Y_e - Y_e \cdot X_e).$$

□

**Theorem 4.20.** *Let  $G$  be a Lie group. Then the tangent bundle  $TG$  of  $G$  is trivial.*

PROOF. Let  $\{X_1, \dots, X_m\}$  be a basis for  $T_eG$ . Then the map  $\psi : TG \rightarrow G \times \mathbb{R}^m$  given by

$$\psi : (p, \sum_{k=1}^m v_k \cdot (X_k^*)_p) \mapsto (p, (v_1, \dots, v_m))$$

is a global bundle chart so the tangent bundle  $TG$  is trivial. □

## Exercises

**Exercise 4.1.** Let  $(M^m, \hat{\mathcal{A}})$  be a smooth manifold and  $(U, x), (V, y)$  be two charts in  $\hat{\mathcal{A}}$  such that  $U \cap V \neq \emptyset$ . Let

$$f = y \circ x^{-1} : x(U \cap V) \rightarrow \mathbb{R}^m$$

be the corresponding transition map. Show that the local frames

$$\left\{ \frac{\partial}{\partial x_i} \mid i = 1, \dots, m \right\} \quad \text{and} \quad \left\{ \frac{\partial}{\partial y_j} \mid j = 1, \dots, m \right\}$$

for  $TM$  on  $U \cap V$  are related by

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^m \frac{\partial(f_j \circ x)}{\partial x_i} \cdot \frac{\partial}{\partial y_j}.$$

**Exercise 4.2.** Let  $m$  be a positive integer and  $\mathbf{SO}(m)$  be the corresponding special orthogonal group.

- (i) Find a basis for the tangent space  $T_e \mathbf{SO}(m)$ ,
- (ii) construct a non-vanishing vector field  $Z \in C^\infty(T\mathbf{SO}(m))$ ,
- (iii) determine all smooth vector fields on  $\mathbf{SO}(2)$ .

**The Hairy Ball Theorem.** Let  $m$  be a positive integer. Then there does not exist a continuous non-vanishing vector field  $X \in C^0(TS^{2m})$  on the even dimensional sphere  $S^{2m}$ .

**Exercise 4.3.** Let  $m$  be a positive integer. Use the Hairy Ball Theorem to prove that the tangent bundles  $TS^{2m}$  of the even-dimensional spheres  $S^{2m}$  are not trivial. Construct a non-vanishing vector field  $X \in C^\infty(TS^{2m+1})$  on the odd-dimensional sphere  $S^{2m+1}$ .

**Exercise 4.4.** Find a proof for Theorem 4.13.

**Exercise 4.5.** Let  $\{\partial/\partial x_k \mid k = 1, \dots, m\}$  be the standard global frame for  $T\mathbb{R}^m$ . Let  $X, Y \in C^\infty(T\mathbb{R}^m)$  be two vector fields given by

$$X = \sum_{k=1}^m \alpha_k \frac{\partial}{\partial x_k} \quad \text{and} \quad Y = \sum_{k=1}^m \beta_k \frac{\partial}{\partial x_k},$$

where  $\alpha_k, \beta_k \in C^\infty(\mathbb{R}^m)$ . Calculate the Lie bracket  $[X, Y]$ .

## Riemannian Manifolds

In this chapter we introduce the important notion of a Riemannian metric on a differentiable manifold. This is the most important example of what is called a tensor field. The metric provides us with an inner product on each tangent space and can be used to measure the length of curves in the manifold. It defines a distance function on the manifold and turns it into a metric space.

Let  $M$  be a smooth manifold,  $C^\infty(M)$  denote the commutative ring of smooth functions on  $M$  and  $C^\infty(TM)$  be the set of smooth vector fields on  $M$  forming a module over  $C^\infty(M)$ . Define  $C_0^\infty(TM) = C^\infty(M)$  and for each  $k \in \mathbb{Z}^+$  let

$$C_k^\infty(TM) = C^\infty(TM) \otimes \cdots \otimes C^\infty(TM)$$

be the  $k$ -fold tensor product of  $C^\infty(TM)$ . A **tensor field**  $B$  on  $M$  of type  $(r, s)$  is a map  $B : C_r^\infty(TM) \rightarrow C_s^\infty(TM)$  satisfying

$$\begin{aligned} & B(X_1 \otimes \cdots \otimes X_{l-1} \otimes (f \cdot Y + g \cdot Z) \otimes X_{l+1} \otimes \cdots \otimes X_r) \\ = & f \cdot B(X_1 \otimes \cdots \otimes X_{l-1} \otimes Y \otimes X_{l+1} \otimes \cdots \otimes X_r) \\ & + g \cdot B(X_1 \otimes \cdots \otimes X_{l-1} \otimes Z \otimes X_{l+1} \otimes \cdots \otimes X_r) \end{aligned}$$

for all  $X_1, \dots, X_r, Y, Z \in C^\infty(TM)$ ,  $f, g \in C^\infty(M)$  and  $l = 1, \dots, r$ . For the rest of this work we shall use the notation  $B(X_1, \dots, X_r)$  for  $B(X_1 \otimes \cdots \otimes X_r)$ .

The next result provides us with the most important property concerning tensor fields. It shows that the value of

$$B(X_1, \dots, X_r)$$

at the point  $p \in M$  only depends on the values of the vector fields  $X_1, \dots, X_r$  at  $p$  and is independent of their values away from  $p$ .

**Proposition 5.1.** *Let  $B : C_r^\infty(TM) \rightarrow C_s^\infty(TM)$  be a tensor field of type  $(r, s)$  and  $p \in M$ . Let  $X_1, \dots, X_r$  and  $Y_1, \dots, Y_r$  be smooth vector fields on  $M$  such that  $(X_k)_p = (Y_k)_p$  for each  $k = 1, \dots, r$ . Then*

$$B(X_1, \dots, X_r)(p) = B(Y_1, \dots, Y_r)(p).$$

PROOF. We shall prove the statement for  $r = 1$ , the rest follows by induction. Put  $X = X_1$  and  $Y = Y_1$  and let  $(U, x)$  be local coordinates on  $M$ . Choose a function  $f \in C^\infty(M)$  such that  $f(p) = 1$ ,

$$\text{support}(f) = \{p \in M \mid f(p) \neq 0\}$$

is contained in  $U$  and define the vector fields  $v_1, \dots, v_m \in C^\infty(TM)$  on  $M$  by

$$(v_k)_q = \begin{cases} f(q) \cdot \left(\frac{\partial}{\partial x_k}\right)_q & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

Then there exist functions  $\rho_k, \sigma_k \in C^\infty(M)$  such that

$$f \cdot X = \sum_{k=1}^m \rho_k v_k \quad \text{and} \quad f \cdot Y = \sum_{k=1}^m \sigma_k v_k.$$

This implies that

$$B(X)(p) = f(p)B(X)(p) = B(f \cdot X)(p) = \sum_{k=1}^m \rho_k(p)B(v_k)(p)$$

and similarly

$$B(Y)(p) = \sum_{k=1}^m \sigma_k(p)B(v_k)(p).$$

The fact that  $X_p = Y_p$  shows that  $\rho_k(p) = \sigma_k(p)$  for all  $k$ . As a direct consequence we see that

$$B(X)(p) = B(Y)(p).$$

□

We shall by  $B_p$  denote the multi-linear restriction of the tensor field  $B$  to the  $r$ -fold tensor product

$$\bigotimes_{l=1}^r T_p M$$

of the vector space  $T_p M$  given by

$$B_p : ((X_1)_p, \dots, (X_r)_p) \mapsto B(X_1, \dots, X_r)(p).$$

**Definition 5.2.** Let  $M$  be a smooth manifold. A **Riemannian metric**  $g$  on  $M$  is a tensor field

$$g : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$$

such that for each  $p \in M$  the restriction

$$g_p = g|_{T_p M \otimes T_p M} : T_p M \otimes T_p M \rightarrow \mathbb{R}$$

with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is an inner product on the tangent space  $T_pM$ . The pair  $(M, g)$  is called a **Riemannian manifold**. The study of Riemannian manifolds is called Riemannian Geometry. Geometric properties of  $(M, g)$  which only depend on the metric  $g$  are called **intrinsic** or metric properties.

The standard inner product on the vector space  $\mathbb{R}^m$  given by

$$\langle u, v \rangle_{\mathbb{R}^m} = \sum_{k=1}^m u_k v_k$$

defines a Riemannian metric on  $\mathbb{R}^m$ . The Riemannian manifold

$$E^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$$

is called the  $m$ -dimensional **Euclidean space**.

**Definition 5.3.** Let  $\gamma : I \rightarrow M$  be a  $C^1$ -curve in  $M$ . Then the **length**  $L(\gamma)$  of  $\gamma$  is defined by

$$L(\gamma) = \int_I \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

By multiplying the Euclidean metric on subsets of  $\mathbb{R}^m$  by a factor we obtain important examples of Riemannian manifolds.

**Example 5.4.** For a positive integer  $m$  equip the real vector space  $\mathbb{R}^m$  with the Riemannian metric  $g$  given by

$$g_x(X, Y) = \frac{4}{(1 + |x|_{\mathbb{R}^m}^2)^2} \langle X, Y \rangle_{\mathbb{R}^m}.$$

The Riemannian manifold  $\Sigma^m = (\mathbb{R}^n, g)$  is called the  $m$ -dimensional **punctured round sphere**. Let  $\gamma : \mathbb{R}^+ \rightarrow \Sigma^m$  be the curve with  $\gamma : t \mapsto (t, 0, \dots, 0)$ . Then the length  $L(\gamma)$  of  $\gamma$  can be determined as follows

$$L(\gamma) = 2 \int_0^\infty \frac{\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}}{1 + |\gamma|^2} dt = 2 \int_0^\infty \frac{dt}{1 + t^2} = 2[\arctan(t)]_0^\infty = \pi.$$

**Example 5.5.** Let  $B_1^m(0)$  be the  $m$ -dimensional open unit ball given by

$$B_1^m(0) = \{x \in \mathbb{R}^m \mid |x|_{\mathbb{R}^m} < 1\}.$$

By the **hyperbolic space**  $H^m$  we mean  $B_1^m(0)$  equipped with the Riemannian metric

$$g_x(X, Y) = \frac{4}{(1 - |x|_{\mathbb{R}^m}^2)^2} \langle X, Y \rangle_{\mathbb{R}^m}.$$

Let  $\gamma : (0, 1) \rightarrow H^m$  be a curve given by  $\gamma : t \mapsto (t, 0, \dots, 0)$ . Then

$$L(\gamma) = 2 \int_0^1 \frac{\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}}{1 - |\gamma|^2} dt = 2 \int_0^1 \frac{dt}{1 - t^2} = [\log(\frac{1+t}{1-t})]_0^1 = \infty$$

As we shall now see a Riemannian manifold  $(M, g)$  has the structure of a metric space  $(M, d)$  in a natural way.

**Proposition 5.6.** *Let  $(M, g)$  be a Riemannian manifold. For two points  $p, q \in M$  let  $C_{pq}$  denote the set of  $C^1$ -curves  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$  and define the function  $d : M \times M \rightarrow \mathbb{R}_0^+$  by*

$$d(p, q) = \inf\{L(\gamma) \mid \gamma \in C_{pq}\}.$$

Then  $(M, d)$  is a metric space i.e. for all  $p, q, r \in M$  we have

- (i)  $d(p, q) \geq 0$ ,
- (ii)  $d(p, q) = 0$  if and only if  $p = q$ ,
- (iii)  $d(p, q) = d(q, p)$ ,
- (iv)  $d(p, q) \leq d(p, r) + d(r, q)$ .

The topology on  $M$  induced by the metric  $d$  is identical to the one  $M$  carries as a topological manifold  $(M, \mathcal{T})$ , see Definition 2.1.

PROOF. See for example: Peter Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998).  $\square$

A Riemannian metric on a differentiable manifold induces a Riemannian metric on any of its submanifolds as follows.

**Definition 5.7.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$ . Then the smooth tensor field  $g : C_2^\infty(TM) \rightarrow C_0^\infty(M)$  given by

$$g(X, Y) : p \mapsto h_p(X_p, Y_p).$$

is a Riemannian metric on  $M$  called the **induced metric** on  $M$  in  $(N, h)$ .

The Euclidean metric  $\langle, \rangle$  on  $\mathbb{R}^n$  induces Riemannian metrics on the following submanifolds.

- (i) the  $m$ -dimensional sphere  $S^m \subset \mathbb{R}^{m+1}$ ,
- (ii) the tangent bundle  $TS^m \subset \mathbb{R}^n$  where  $n = 2m + 2$ ,
- (iii) the  $m$ -dimensional torus  $T^m \subset \mathbb{R}^n$ , where  $n = 2m$
- (iv) the  $m$ -dimensional real projective space  $\mathbb{R}P^m \subset \text{Sym}(\mathbb{R}^{m+1}) \subset \mathbb{R}^n$  where  $n = (m + 2)(m + 1)/2$ .

The set of complex  $m \times m$  matrices  $\mathbb{C}^{m \times m}$  carries a natural Euclidean metric  $\langle, \rangle$  given by

$$\langle Z, W \rangle = \text{Re}\{\text{trace}(\bar{Z}^t \cdot W)\}.$$



This induces metrics on the submanifolds of  $\mathbb{C}^{m \times m}$  such as  $\mathbb{R}^{m \times m}$  and the classical Lie groups  $\mathbf{GL}(\mathbb{C}^m)$ ,  $\mathbf{SL}(\mathbb{C}^m)$ ,  $\mathbf{U}(m)$ ,  $\mathbf{SU}(m)$ ,  $\mathbf{GL}(\mathbb{R}^m)$ ,  $\mathbf{SL}(\mathbb{R}^m)$ ,  $\mathbf{O}(m)$  and  $\mathbf{SO}(m)$ .

Our next important step is to prove that every differentiable manifold  $M$  can be equipped with a Riemannian metric  $g$ . For this we need the following fact from topology.

**Fact 5.8.** *Every locally compact Hausdorff space with countable basis is paracompact.*

**Corollary 5.9.** *Let  $(M, \hat{\mathcal{A}})$  be a topological manifold. Let the collection  $(U_\alpha)_{\alpha \in I}$  be an open covering of  $M$  such that for each  $\alpha \in I$  the pair  $(U_\alpha, \psi_\alpha)$  is a chart on  $M$ . Then there exists*

- (i) *a locally finite open refinement  $(W_\beta)_{\beta \in J}$  such that for all  $\beta \in J$ ,  $W_\beta$  is an open neighbourhood for a chart  $(W_\beta, \psi_\beta) \in \hat{\mathcal{A}}$ , and*
- (ii) *a partition of unity  $(f_\beta)_{\beta \in J}$  such that  $\text{support}(f_\beta) \subset W_\beta$ .*

**Theorem 5.10.** *Let  $(M^m, \hat{\mathcal{A}})$  be a differentiable manifold. Then there exists a Riemannian metric  $g$  on  $M$ .*

PROOF. For each point  $p \in M$  let  $(U_p, \phi_p)$  be a chart such that  $p \in U_p$ . Then  $(U_p)_{p \in M}$  is an open covering as in Corollary 5.9. Let  $(W_\beta)_{\beta \in J}$  be a locally finite open refinement,  $(W_\beta, x_\beta)$  be charts on  $M$  and  $(f_\beta)_{\beta \in J}$  be a partition of unity such that  $\text{support}(f_\beta)$  is contained in  $W_\beta$ . Let  $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$  be the Euclidean metric on  $\mathbb{R}^m$ . Then for  $\beta \in J$  define  $g_\beta : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  by

$$g_\beta\left(\frac{\partial}{\partial x_k^\beta}, \frac{\partial}{\partial x_l^\beta}\right)(p) = \begin{cases} f_\beta(p) \cdot \langle e_k, e_l \rangle_{\mathbb{R}^m} & \text{if } p \in W_\beta \\ 0 & \text{if } p \notin W_\beta \end{cases}$$

Then  $g : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  given by  $g = \sum_{\beta \in J} g_\beta$  is a Riemannian metric on  $M$ .  $\square$

**Definition 5.11.** Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds. A map  $\phi : (M, g) \rightarrow (N, h)$  is said to be **conformal** if there exists a function  $\lambda : M \rightarrow \mathbb{R}$  such that

$$e^{\lambda(p)} g_p(X_p, Y_p) = h_{\phi(p)}(d\phi_p(X_p), d\phi_p(Y_p)),$$

for all  $X, Y \in C^\infty(TM)$  and  $p \in M$ . The function  $e^\lambda$  is called the **conformal factor** of  $\phi$ . A conformal map with  $\lambda \equiv 0$  is said to be **isometric**. An isometric diffeomorphism is called an **isometry**.

On the standard unit sphere  $S^m$  we have an action  $\mathbf{O}(m) \times S^m \rightarrow S^m$  of the orthogonal group  $\mathbf{O}(m)$  given by

$$(A, x) \mapsto A \cdot x$$

where  $\cdot$  is the standard matrix multiplication. The following shows that the  $\mathbf{O}(m)$ -action on  $S^m$  is isometric

$$\langle Ax, Ay \rangle = x^t A^t Ay = x^t y = \langle x, y \rangle.$$

**Example 5.12.** Equip the orthogonal group  $\mathbf{O}(m)$  as a submanifold of  $\mathbb{R}^{m \times m}$  with the induced metric given by

$$\langle X, Y \rangle = \text{trace}(X^t \cdot Y).$$

For  $x \in \mathbf{O}(m)$  the left translation  $L_x : \mathbf{O}(m) \rightarrow \mathbf{O}(m)$  by  $x$  is given by  $L_x : y \mapsto xy$ . The tangent space  $T_y \mathbf{O}(m)$  of  $\mathbf{O}(m)$  at  $y$  is

$$T_y \mathbf{O}(m) = \{y \cdot X \mid X + X^t = 0\}$$

and the differential  $(dL_x)_y : T_y \mathbf{O}(m) \rightarrow T_{xy} \mathbf{O}(m)$  of  $L_x$  is given by

$$(dL_x)_y : yX \mapsto xyX.$$

We then have

$$\begin{aligned} \langle (dL_x)_y(yX), (dL_x)_y(yY) \rangle_{xy} &= \text{trace}((xyX)^t xyY) \\ &= \text{trace}(X^t y^t x^t xyY) \\ &= \text{trace}(yX)^t (yY). \\ &= \langle yX, yY \rangle_y. \end{aligned}$$

This shows that the left translation  $L_x : \mathbf{O}(m) \rightarrow \mathbf{O}(m)$  is an isometry for each  $x \in \mathbf{O}(m)$ .

**Definition 5.13.** Let  $G$  be a Lie group. A Riemannian metric  $g$  on  $G$  is said to be left invariant if for each  $x \in G$  the left translation  $L_x : G \rightarrow G$  is an isometry.

As for the orthogonal group  $\mathbf{O}(m)$  an inner product on the tangent space at the neutral element of any Lie group can be transported via the left translations to obtain a left invariant Riemannian metric on the group.

**Proposition 5.14.** Let  $G$  be a Lie group and  $\langle \cdot, \cdot \rangle_e$  be an inner product on the tangent space  $T_e G$  at the neutral element  $e$ . Then for each  $x \in G$  the bilinear map  $g_x(\cdot, \cdot) : T_x G \times T_x G \rightarrow \mathbb{R}$  with

$$g_x(X_x, Y_x) = \langle dL_{x^{-1}}(X_x), dL_{x^{-1}}(Y_x) \rangle_e$$

is an inner product on the tangent space  $T_x G$ . The smooth tensor field  $g : C_2^\infty(TG) \rightarrow C_0^\infty(G)$  given by

$$g : (X, Y) \mapsto (g(X, Y) : x \mapsto g_x(X_x, Y_x))$$

is a left invariant Riemannian metric on  $G$ .

PROOF. See Exercise 5.4. □

We shall now equip the real projective space  $\mathbb{R}P^m$  with a Riemannian metric.

**Example 5.15.** Let  $S^m$  be the unit sphere in  $\mathbb{R}^{m+1}$  and  $\text{Sym}(\mathbb{R}^{m+1})$  be the linear space of symmetric real  $(m+1) \times (m+1)$  matrices equipped with the metric  $g$  given by

$$g(A, B) = \text{trace}(A^t \cdot B)/8.$$

As in Example 3.21 we define a map  $\phi : S^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  by

$$\phi : p \mapsto (\rho_p : q \mapsto 2\langle q, p \rangle p - q).$$

Let  $\alpha, \beta : \mathbb{R} \rightarrow S^m$  be two curves such that  $\alpha(0) = p = \beta(0)$  and put  $a = \alpha'(0)$ ,  $b = \beta'(0)$ . Then for  $\gamma \in \{\alpha, \beta\}$  we have

$$d\phi_p(\gamma'(0)) = (q \mapsto 2\langle q, \gamma'(0) \rangle p + 2\langle q, p \rangle \gamma'(0)).$$

If  $\mathcal{B}$  is an orthonormal basis for  $\mathbb{R}^{m+1}$ , then

$$\begin{aligned} g(d\phi_p(a), d\phi_p(b)) &= \text{trace}(d\phi_p(a)^t \cdot d\phi_p(b))/8 \\ &= \sum_{q \in \mathcal{B}} \langle \langle q, a \rangle p + \langle q, p \rangle a, \langle q, b \rangle p + \langle q, p \rangle b \rangle / 2 \\ &= \sum_{q \in \mathcal{B}} \{ \langle p, p \rangle \langle a, q \rangle \langle q, b \rangle + \langle a, b \rangle \langle p, q \rangle \langle p, q \rangle \} / 2 \\ &= \{ \langle a, b \rangle + \langle a, b \rangle \} / 2 \\ &= \langle a, b \rangle \end{aligned}$$

This proves that the immersion  $\phi$  is isometric. In Example 3.21 we have seen that the image  $\phi(S^m)$  can be identified with the real projective space  $\mathbb{R}P^m$ . This inherits the induced metric from  $\mathbb{R}^{(m+1) \times (m+1)}$  and the map  $\phi : S^m \rightarrow \mathbb{R}P^m$  is what is called an isometric double cover of  $\mathbb{R}P^m$ .

Long before John Nash became famous in Hollywood he proved the next remarkable result in his paper *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20-63. It implies that every Riemannian manifold can be realized as a submanifold of a Euclidean space. The original proof of Nash was later simplified, see for example Matthias Gunther, *On the perturbation problem associated to isometric embeddings of Riemannian manifolds*, Annals of Global Analysis and Geometry **7** (1989), 69-77.

**Deep Result 5.16.** For  $3 \leq r \leq \infty$  let  $(M, g)$  be a Riemannian  $C^r$ -manifold. Then there exists an isometric  $C^r$ -embedding of  $(M, g)$  into a Euclidean space  $\mathbb{R}^n$ .

We shall now see that parametrizations can be very useful tools for studying the intrinsic geometry of a Riemannian manifold  $(M, g)$ . Let  $p$  be a point of  $M$  and  $\hat{\psi} : U \rightarrow M$  be a local parametrization of  $M$  with  $q \in U$  and  $\hat{\psi}(q) = p$ . The differential  $d\hat{\psi}_q : T_q\mathbb{R}^m \rightarrow T_pM$  is bijective so there exist neighbourhoods  $U_q$  of  $q$  and  $U_p$  of  $p$  such that the restriction  $\psi = \hat{\psi}|_{U_q} : U_q \rightarrow U_p$  is a diffeomorphism. On  $U_q$  we have the canonical frame  $\{e_1, \dots, e_m\}$  for  $TU_q$  so  $\{d\psi(e_1), \dots, d\psi(e_m)\}$  is a local frame for  $TM$  over  $U_p$ . We then define the pull-back metric  $\tilde{g} = \psi^*g$  on  $U_q$  by

$$\tilde{g}(e_k, e_l) = g(d\psi(e_k), d\psi(e_l)).$$

Then  $\psi : (U_q, \tilde{g}) \rightarrow (U_p, g)$  is an isometry so the intrinsic geometry of  $(U_q, \tilde{g})$  and that of  $(U_p, g)$  are exactly the same.

**Example 5.17.** Let  $G$  be one of the classical Lie groups and  $e$  be the neutral element of  $G$ . Let  $\{X_1, \dots, X_m\}$  be a basis for the Lie algebra  $\mathfrak{g}$  of  $G$ . For  $x \in G$  define  $\psi_x : \mathbb{R}^m \rightarrow G$  by

$$\psi_x : (t_1, \dots, t_m) \mapsto L_x\left(\prod_{k=1}^m \text{Exp}(t_k X_k)\right)$$

where  $L_x : G \rightarrow G$  is the left-translation given by  $L_x(y) = xy$ . Then

$$(d\psi_x)_0(e_k) = X_k(x)$$

for all  $k$ . This means that the differential  $(d\psi_x)_0 : T_0\mathbb{R}^m \rightarrow T_xG$  is an isomorphism so there exist open neighbourhoods  $U_0$  of  $0$  and  $U_x$  of  $x$  such that the restriction of  $\psi$  to  $U_0$  is bijective onto its image  $U_x$  and hence a local parametrization of  $G$  around  $x$ .

We shall now study the normal bundle of a submanifold of a given Riemannian manifold. This is an important example of the notion of a vector bundle over a manifold.

**Definition 5.18.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$ . For a point  $p \in M$  we define the **normal space**  $N_pM$  of  $M$  at  $p$  by

$$N_pM = \{v \in T_pN \mid h_p(v, w) = 0 \text{ for all } w \in T_pM\}.$$

For all  $p$  we have the orthogonal decomposition

$$T_pN = T_pM \oplus N_pM.$$

The **normal bundle** of  $M$  in  $N$  is defined by

$$NM = \{(p, v) \mid p \in M, v \in N_pM\}.$$

**Example 5.19.** Let  $S^m$  be the unit sphere in  $\mathbb{R}^{m+1}$  equipped with its standard Euclidean metric  $\langle \cdot, \cdot \rangle$ . If  $p \in S^m$  then the tangent space  $T_p S^m$  of  $S^m$  at  $p$  is

$$T_p S^m = \{v \in \mathbb{R}^{m+1} \mid \langle v, p \rangle = 0\}$$

so the normal space  $N_p S^m$  of  $S^m$  at  $p$  satisfies

$$N_p S^m = \{t \cdot p \in \mathbb{R}^{m+1} \mid t \in \mathbb{R}\}.$$

This shows that the normal bundle  $NS^m$  of  $S^m$  in  $\mathbb{R}^{m+1}$  is given by

$$NS^m = \{(p, t \cdot p) \in \mathbb{R}^{2m+2} \mid p \in S^m, t \in \mathbb{R}\}.$$

**Theorem 5.20.** Let  $(N^n, h)$  be a Riemannian manifold and  $M^m$  be a smooth submanifold of  $N$ . Then the normal bundle  $(NM, M, \pi)$  is a smooth  $(n - m)$ -dimensional vector bundle over  $M$ .

PROOF. See Exercise 5.6. □

We shall now determine the normal bundle  $NO(m)$  of the orthogonal group  $O(m)$  as a submanifold of  $\mathbb{R}^{m \times m}$ .

**Example 5.21.** The orthogonal group  $O(m)$  is a subset of the linear space  $\mathbb{R}^{m \times m}$  equipped with the Riemannian metric

$$\langle X, Y \rangle = \text{trace}(X^t \cdot Y)$$

inducing a left invariant metric on  $O(m)$ . We have already seen that the tangent space  $T_e O(m)$  of  $O(m)$  at the neutral element  $e$  is

$$T_e O(m) = \{X \in \mathbb{R}^{m \times m} \mid X + X^t = 0\}$$

and that the tangent bundle  $TO(m)$  of  $O(m)$  is given by

$$TO(m) = \{(x, xX) \mid x \in O(m), X \in T_e O(m)\}.$$

The space  $\mathbb{R}^{m \times m}$  of real  $m \times m$  matrices has a linear decomposition

$$\mathbb{R}^{m \times m} = \text{Sym}(\mathbb{R}^m) \oplus T_e O(m)$$

and every element  $X \in \mathbb{R}^{m \times m}$  can be decomposed  $X = X^\top + X^\perp$  in its symmetric and skew-symmetric parts given by

$$X^\top = (X + X^t)/2 \quad \text{and} \quad X^\perp = (X - X^t)/2.$$

If  $X \in T_e O(m)$  and  $Y \in \text{Sym}(\mathbb{R}^m)$  then

$$\begin{aligned} \langle X, Y \rangle &= \text{trace}(X^t Y) \\ &= \text{trace}(Y^t X) \\ &= \text{trace}(XY^t) \\ &= \text{trace}(-X^t Y) \\ &= -\langle X, Y \rangle. \end{aligned}$$

This means that the normal bundle  $N\mathbf{O}(m)$  of  $\mathbf{O}(m)$  in  $\mathbb{R}^{m \times m}$  is given by

$$N\mathbf{O}(m) = \{(x, xY) \mid x \in \mathbf{O}(m), Y \in \text{Sym}(\mathbb{R}^m)\}.$$

A given Riemannian metric  $g$  on  $M$  can be used to construct a family of natural metrics on the tangent bundle  $TM$  of  $M$ . The best known such examples are the Sasaki and Cheeger-Gromoll metrics. For a detailed survey on the geometry of tangent bundles equipped with these metrics we recommend the paper S. Gudmundsson, E. Kappos, *On the geometry of tangent bundles*, Expo. Math. **20** (2002), 1-41.

## Exercises

**Exercise 5.1.** Let  $m$  be a positive integer and  $\phi : \mathbb{R}^m \rightarrow \mathbb{C}^m$  be the standard parametrization of the  $m$ -dimensional torus  $T^m$  in  $\mathbb{C}^m$  given by  $\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$ . Prove that  $\phi$  is an isometric parametrization.

**Exercise 5.2.** Let  $m$  be a positive integer and

$$\pi_m : (S^m - \{(1, 0, \dots, 0)\}, \langle, \rangle_{\mathbb{R}^{m+1}}) \rightarrow (\mathbb{R}^m, \frac{4}{(1 + |x|^2)^2} \langle, \rangle_{\mathbb{R}^m})$$

be the **stereographic projection** given by

$$\pi_m : (x_0, \dots, x_m) \mapsto \frac{1}{1 - x_0} (x_1, \dots, x_m).$$

Prove that  $\pi_m$  is an isometry.

**Exercise 5.3.** Let  $B_1^2(0)$  be the open unit disk in the complex plane equipped with the hyperbolic metric given by

$$g(X, Y) = 4/(1 - |z|^2)^2 \langle X, Y \rangle_{\mathbb{R}^2}.$$

Prove that the map

$$\pi : B_1^2(0) \rightarrow (\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}, \frac{1}{\text{Im}(z)^2} \langle, \rangle_{\mathbb{R}^2})$$

with

$$\pi : z \mapsto \frac{i + z}{1 + iz}$$

is an isometry.

**Exercise 5.4.** Find a proof for Proposition 5.14.

**Exercise 5.5.** Let  $m$  be a positive integer and  $\mathbf{GL}(\mathbb{R}^m)$  be the corresponding real general linear group. Let  $g, h$  be two Riemannian metrics on  $\mathbf{GL}(\mathbb{R}^m)$  defined by

$$g_x(xZ, xW) = \text{trace}((xZ)^t \cdot xW), \quad h_x(xZ, xW) = \text{trace}(Z^t \cdot W).$$

Further let  $\hat{g}, \hat{h}$  be the induced metrics on the special linear group  $\mathbf{SL}(\mathbb{R}^m)$  as a subset of  $\mathbf{GL}(\mathbb{R}^m)$ .

- (i) Which of the metrics  $g, h, \hat{g}, \hat{h}$  are left-invariant?
- (ii) Determine the normal space  $N_e \mathbf{SL}(\mathbb{R}^m)$  of  $\mathbf{SL}(\mathbb{R}^m)$  in  $\mathbf{GL}(\mathbb{R}^m)$  with respect to  $g$
- (iii) Determine the normal bundle  $N \mathbf{SL}(\mathbb{R}^m)$  of  $\mathbf{SL}(\mathbb{R}^m)$  in  $\mathbf{GL}(\mathbb{R}^m)$  with respect to  $h$ .

**Exercise 5.6.** Find a proof for Theorem 5.20.





## The Levi-Civita Connection

In this chapter we introduce the notion of a connection in a smooth vector bundle. We study, in detail, the important case of the tangent bundle  $(TM, M, \pi)$  of a smooth Riemannian manifold  $(M, g)$ . We introduce the Levi-Civita connection on  $TM$  and prove that this is the unique connection on the tangent bundle which is both metric and 'torsion free'. We deduce an explicit formula for the Levi-Civita connection for certain Lie groups. Finally we give an example of a connection in the normal bundle of a submanifold of a Riemannian manifold and study its properties.

On the  $m$ -dimensional real vector space  $\mathbb{R}^m$  we have the well-known differential operator

$$\partial : C^\infty(T\mathbb{R}^m) \times C^\infty(T\mathbb{R}^m) \rightarrow C^\infty(T\mathbb{R}^m)$$

mapping a pair of vector fields  $X, Y$  on  $\mathbb{R}^m$  to the directional derivative  $\partial_X Y$  of  $Y$  in the direction of  $X$  given by

$$(\partial_X Y)(x) = \lim_{t \rightarrow 0} \frac{Y(x + tX(x)) - Y(x)}{t}.$$

The most fundamental properties of the operator  $\partial$  are expressed by the following. If  $\lambda, \mu \in \mathbb{R}$ ,  $f, g \in C^\infty(\mathbb{R}^m)$  and  $X, Y, Z \in C^\infty(T\mathbb{R}^m)$  then

- (i)  $\partial_X(\lambda \cdot Y + \mu \cdot Z) = \lambda \cdot \partial_X Y + \mu \cdot \partial_X Z$ ,
- (ii)  $\partial_X(f \cdot Y) = \partial_X(f) \cdot Y + f \cdot \partial_X Y$ ,
- (iii)  $\partial(f \cdot X + g \cdot Y)Z = f \cdot \partial_X Z + g \cdot \partial_Y Z$ .

Further well-known properties of the differential operator  $\partial$  are given by the next result.

**Proposition 6.1.** *Let the real vector space  $\mathbb{R}^m$  be equipped with the standard Euclidean metric  $\langle \cdot, \cdot \rangle$  and  $X, Y, Z \in C^\infty(T\mathbb{R}^m)$  be smooth vector fields. Then*

- (iv)  $\partial_X Y - \partial_Y X = [X, Y]$ ,
- (v)  $\partial_X(\langle Y, Z \rangle) = \langle \partial_X Y, Z \rangle + \langle Y, \partial_X Z \rangle$ .

We shall generalize the operator  $\partial$  on the Euclidean space to any Riemannian manifold  $(M, g)$ . First we define the concept of a connection in a smooth vector bundle.

**Definition 6.2.** Let  $(E, M, \pi)$  be a smooth vector bundle over  $M$ . A **connection** on  $(E, M, \pi)$  is a map  $\hat{\nabla} : C^\infty(TM) \times C^\infty(E) \rightarrow C^\infty(E)$  such that

- (i)  $\hat{\nabla}_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \hat{\nabla}_X v + \mu \cdot \hat{\nabla}_X w$ ,
- (ii)  $\hat{\nabla}_X(f \cdot v) = X(f) \cdot v + f \cdot \hat{\nabla}_X v$ ,
- (iii)  $\hat{\nabla}(f \cdot X + g \cdot Y)v = f \cdot \hat{\nabla}_X v + g \cdot \hat{\nabla}_Y v$ .

for all  $\lambda, \mu \in \mathbb{R}$ ,  $X, Y \in C^\infty(TM)$ ,  $v, w \in C^\infty(E)$  and  $f, g \in C^\infty(M)$ . A section  $v \in C^\infty(E)$  is said to be **parallel** with respect to the connection  $\hat{\nabla}$  if  $\hat{\nabla}_X v = 0$  for all  $X \in C^\infty(TM)$ .

**Definition 6.3.** Let  $M$  be a smooth manifold and  $\hat{\nabla}$  be a connection on the tangent bundle  $(TM, M, \pi)$ . Then the **torsion** of  $\hat{\nabla} : C_2^\infty(TM) \rightarrow C_1^\infty(TM)$  is defined by

$$T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y],$$

where  $[,]$  is the Lie bracket on  $C^\infty(TM)$ . A connection  $\hat{\nabla}$  on the tangent bundle  $(TM, M, \pi)$  is said to be **torsion-free** if the corresponding torsion  $T$  vanishes i.e.

$$[X, Y] = \hat{\nabla}_X Y - \hat{\nabla}_Y X$$

for all  $X, Y \in C^\infty(TM)$ . If  $g$  is a Riemannian metric on  $M$  then  $\hat{\nabla}$  is said to be **metric** or **compatible with  $g$**  if

$$X(g(Y, Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z)$$

for all  $X, Y, Z \in C^\infty(TM)$ .

Let us now assume that  $\nabla$  is a metric and torsion-free connection on the tangent bundle  $TM$  of a differentiable manifold  $M$ . Then it is easily seen that the following equations hold

$$g(\nabla_X Y, Z) = X(g(Y, Z)) - g(Y, \nabla_X Z),$$

$$\begin{aligned} g(\nabla_X Y, Z) &= g([X, Y], Z) + g(\nabla_Y X, Z) \\ &= g([X, Y], Z) + Y(g(X, Z)) - g(X, \nabla_Y Z), \end{aligned}$$

$$\begin{aligned} 0 &= -Z(g(X, Y)) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\ &= -Z(g(X, Y)) + g(\nabla_X Z + [Z, X], Y) + g(X, \nabla_Y Z - [Y, Z]). \end{aligned}$$

By adding these relations we yield

$$2 \cdot g(\nabla_X Y, Z) = \{X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z])\}.$$

If  $\{E_1, \dots, E_m\}$  is a local orthonormal frame for the tangent bundle then

$$\nabla_X Y = \sum_{k=1}^m g(\nabla_X Y, E_k) E_k.$$

This implies that there exists *at most* one metric and torsion free connection on the tangent bundle.

**Definition 6.4.** Let  $(M, g)$  be a Riemannian manifold then the map  $\nabla : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$  given by

$$g(\nabla_X Y, Z) = \frac{1}{2} \{X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z])\}.$$

is called the **Levi-Civita connection** on  $M$ .

It should be noted that the Levi-Civita connection is an intrinsic object on  $(M, g)$  only depending on the differentiable structure of the manifold and its Riemannian metric.

**Proposition 6.5.** *Let  $(M, g)$  be a Riemannian manifold. Then the Levi-Civita connection  $\nabla$  is a connection on the tangent bundle  $TM$  of  $M$ .*

**PROOF.** It follows from Definition 3.5, Theorem 4.13 and the fact that  $g$  is a tensor field that

$$g(\nabla_X(\lambda \cdot Y_1 + \mu \cdot Y_2), Z) = \lambda \cdot g(\nabla_X Y_1, Z) + \mu \cdot g(\nabla_X Y_2, Z)$$

and

$$g(\nabla_{Y_1 + Y_2} X, Z) = g(\nabla_{Y_1} X, Z) + g(\nabla_{Y_2} X, Z)$$

for all  $\lambda, \mu \in \mathbb{R}$  and  $X, Y_1, Y_2, Z \in C^\infty(TM)$ . Furthermore we have for all  $f \in C^\infty(M)$

$$\begin{aligned} & g(\nabla_X f Y, Z) \\ &= \frac{1}{2} \{X(f \cdot g(Y, Z)) + f \cdot Y(g(Z, X)) - Z(f \cdot g(X, Y)) \\ &\quad + g(Z, [X, f \cdot Y]) + f \cdot g(Y, [Z, X]) - g(X, [f \cdot Y, Z])\} \\ &= \frac{1}{2} \{X(f) \cdot g(Y, Z) + f \cdot X(g(Y, Z)) + f \cdot Y(g(Z, X)) \\ &\quad - Z(f) \cdot g(X, Y) - f \cdot Z(g(X, Y)) + g(Z, X(f) \cdot Y + f \cdot [X, Y])\} \end{aligned}$$

$$\begin{aligned}
& + f \cdot g(Y, [Z, X]) - g(X, -Z(f) \cdot Y + f \cdot [Y, Z]) \} \\
= & X(f) \cdot g(Y, Z) + f \cdot g(\nabla_X Y, Z) \\
= & g(X(f) \cdot Y + f \cdot \nabla_X Y, Z)
\end{aligned}$$

and

$$\begin{aligned}
& g(\nabla_f \cdot X^Y, Z) \\
= & \frac{1}{2} \{ f \cdot X(g(Y, Z)) + Y(f \cdot g(Z, X)) - Z(f \cdot g(X, Y)) \\
& + g(Z, [f \cdot X, Y]) + g(Y, [Z, f \cdot X]) - f \cdot g(X, [Y, Z]) \} \\
= & \frac{1}{2} \{ f \cdot X(g(Y, Z)) + Y(f) \cdot g(Z, X) + f \cdot Y(g(Z, X)) \\
& - Z(f) \cdot g(X, Y) - f \cdot Z(g(X, Y)) \\
& + g(Z, -Y(f) \cdot X) + g(Z, f \cdot [X, Y]) + g(Y, Z(f) \cdot X) \\
& f \cdot g(Y, [Z, X]) - f \cdot g(X, [Y, Z]) \} \\
= & f \cdot g(\nabla_X Y, Z).
\end{aligned}$$

This proves that  $\nabla$  is a connection on the tangent bundle  $(TM, M, \pi)$ .  $\square$

The next result is called the fundamental theorem of Riemannian geometry.

**Theorem 6.6.** *Let  $(M, g)$  be a Riemannian manifold. Then the Levi-Civita connection is a unique metric and torsion free connection on the tangent bundle  $(TM, M, \pi)$ .*

PROOF. The difference  $g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$  equals twice the skew-symmetric part (w.r.t the pair  $(X, Y)$ ) of the right hand side of the equation in Definition 6.4. This is the same as

$$= \frac{1}{2} \{ g(Z, [X, Y]) - g(Z, [Y, X]) \} = g(Z, [X, Y]).$$

This proves that the Levi-Civita connection is torsion-free.

The sum  $g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$  equals twice the symmetric part (w.r.t the pair  $(Y, Z)$ ) on the right hand side of Definition 6.4. This is exactly

$$= \frac{1}{2} \{ X(g(Y, Z)) + X(g(Z, Y)) \} = X(g(Y, Z)).$$

This shows that the Levi-Civita connection is compatible with the Riemannian metric  $g$  on  $M$ .  $\square$

**Definition 6.7.** Let  $G$  be a Lie group. For a left invariant vector field  $Z \in \mathfrak{g}$  we define the map  $\text{ad}(Z) : \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$\text{ad}(Z) : X \mapsto [Z, X].$$

**Proposition 6.8.** Let  $(G, g)$  be a Lie group equipped with a left invariant metric. Then the Levi-Civita connection  $\nabla$  satisfies

$$g(\nabla_X Y, Z) = \frac{1}{2} \{g(Z, [X, Y]) + g(Y, \text{ad}(Z)(X)) + g(X, \text{ad}(Z)(Y))\}$$

for all  $X, Y, Z \in \mathfrak{g}$ . In particular, if for all  $Z \in \mathfrak{g}$  the map  $\text{ad}(Z)$  is skew symmetric with respect to  $g$  then

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

PROOF. See Exercise 6.2.  $\square$

**Proposition 6.9.** Let  $G$  be one of the compact classical Lie groups  $\mathbf{O}(m)$ ,  $\mathbf{SO}(m)$ ,  $\mathbf{U}(m)$  or  $\mathbf{SU}(m)$  equipped with the metric

$$g(Z, W) = \text{Re}(\text{trace}(\bar{Z}^t \cdot W)).$$

Then for each  $X \in \mathfrak{g}$  the operator  $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$  is skew symmetric.

PROOF. See Exercise 6.3.  $\square$

**Example 6.10.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . Further let  $(U, x)$  be local coordinates on  $M$  and put  $X_i = \partial/\partial x_i \in C^\infty(TU)$ . Then  $\{X_1, \dots, X_m\}$  is a local frame of  $TM$  on  $U$ . For  $(U, x)$  we define the **Christoffel symbols**  $\Gamma_{ij}^k : U \rightarrow \mathbb{R}$  of the connection  $\nabla$  with respect to  $(U, x)$  by

$$\sum_{k=1}^m \Gamma_{ij}^k X_k = \nabla_{X_i} X_j.$$

On the subset  $x(U)$  of  $\mathbb{R}^m$  we define the metric  $\tilde{g}$  by

$$\tilde{g}(e_i, e_j) = g_{ij} = g(X_i, X_j).$$

The differential  $dx$  is bijective so Proposition 4.15 implies that

$$dx([X_i, X_j]) = [dx(X_i), dx(X_j)] = [e_i, e_j] = 0$$

and hence  $[X_i, X_j] = 0$ . From the definition of the Levi-Civita connection we now yield

$$\begin{aligned} \sum_{k=1}^m \Gamma_{ij}^k g_{kl} &= \left\langle \sum_{k=1}^m \Gamma_{ij}^k X_k, X_l \right\rangle \\ &= \langle \nabla_{X_i} X_j, X_l \rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\{X_i\langle X_j, X_l \rangle + X_j\langle X_l, X_i \rangle - X_l\langle X_i, X_j \rangle\} \\
&= \frac{1}{2}\left\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right\}.
\end{aligned}$$

If  $g^{kl} = (g^{-1})_{kl}$  then

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^m g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\}.$$

**Definition 6.11.** Let  $N$  be a smooth manifold,  $M$  be a submanifold of  $N$  and  $\tilde{X} \in C^\infty(TM)$  be a vector field on  $M$ . Let  $U$  be an open subset of  $N$  such that  $U \cap M \neq \emptyset$ . A **local extension** of  $\tilde{X}$  to  $U$  is a vector field  $X \in C^\infty(TU)$  such that  $\tilde{X}_p = X_p$  for all  $p \in M$ . If  $U = N$  then  $X$  is called a **global extension**.

**Fact 6.12.** *Every vector field  $\tilde{X} \in C^\infty(TM)$  has a global extension  $X \in C^\infty(TN)$ .*

Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold equipped with the induced metric  $g$ . Let  $X \in C^\infty(TN)$  be a vector field on  $N$  and  $\tilde{X} = X|_M : M \rightarrow TN$  be the restriction of  $X$  to  $M$ . Note that  $\tilde{X}$  is not necessarily an element of  $C^\infty(TM)$  i.e. a vector field on the submanifold  $M$ . For each  $p \in M$  the tangent vector  $\tilde{X}_p \in T_pN$  can be decomposed

$$\tilde{X}_p = \tilde{X}_p^\top + \tilde{X}_p^\perp$$

in a unique way into its tangential part  $(\tilde{X}_p)^\top \in T_pM$  and its normal part  $(\tilde{X}_p)^\perp \in N_pM$ . For this we write  $\tilde{X} = \tilde{X}^\top + \tilde{X}^\perp$ .

Let  $\tilde{X}, \tilde{Y} \in C^\infty(TM)$  be vector fields on  $M$  and  $X, Y \in C^\infty(TN)$  be their extensions to  $N$ . If  $p \in M$  then  $(\nabla_X Y)_p$  only depends on the value  $X_p = \tilde{X}_p$  and the value of  $Y$  along some curve  $\gamma : (-\epsilon, \epsilon) \rightarrow N$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X_p = \tilde{X}_p$ . For this see Remark 7.3. Since  $X_p \in T_pM$  we may choose the curve  $\gamma$  such that the image  $\gamma((-\epsilon, \epsilon))$  is contained in  $M$ . Then  $\tilde{Y}_{\gamma(t)} = Y_{\gamma(t)}$  for  $t \in (-\epsilon, \epsilon)$ . This means that  $(\nabla_X Y)_p$  only depends on  $\tilde{X}_p$  and the value of  $\tilde{Y}$  along  $\gamma$ , hence independent of the way  $\tilde{X}$  and  $\tilde{Y}$  are extended. This shows that the following maps  $\tilde{\nabla}$  and  $B$  are well defined.

**Definition 6.13.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold equipped with the induced metric  $g$ . Then we define

$$\tilde{\nabla} : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$$

by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^\top,$$

where  $X, Y \in C^\infty(TN)$  are any extensions of  $\tilde{X}, \tilde{Y}$ . Furthermore let

$$B : C_2^\infty(TM) \rightarrow C^\infty(NM)$$

be given by

$$B(\tilde{X}, \tilde{Y}) = (\nabla_X Y)^\perp.$$

It is easily proved that  $B$  is symmetric and hence tensorial in both its arguments, see Exercise 6.6. The operator  $B$  is called the **second fundamental form** of  $M$  in  $(N, h)$ .

**Theorem 6.14.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$  with the induced metric  $g$ . Then  $\tilde{\nabla}$  is the Levi-Civita connection of the submanifold  $(M, g)$ .*

PROOF. See Exercise 6.7. □

The Levi-Civita connection on  $(N, h)$  induces a metric connection  $\bar{\nabla}$  on the normal bundle  $NM$  of  $M$  in  $N$  as follows.

**Proposition 6.15.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Let  $X, Y \in C^\infty(TN)$  be vector fields extending  $\tilde{X} \in C^\infty(TM)$  and  $\tilde{Y} \in C^\infty(NM)$ . Then the map  $\bar{\nabla} : C^\infty(TM) \times C^\infty(NM) \rightarrow C^\infty(NM)$  given by*

$$\bar{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^\perp$$

is a well-defined connection on the normal bundle  $NM$  satisfying

$$\tilde{X}(g(\tilde{Y}, \tilde{Z})) = g(\bar{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) + g(\tilde{Y}, \bar{\nabla}_{\tilde{X}}\tilde{Z})$$

for all  $\tilde{X} \in C^\infty(TM)$  and  $\tilde{Y}, \tilde{Z} \in C^\infty(NM)$ .

PROOF. See Exercise 6.8. □

## Exercises

**Exercise 6.1.** Let  $M$  be a smooth manifold and  $\hat{\nabla}$  be a connection on the tangent bundle  $(TM, M, \pi)$ . Prove that the torsion  $T : C_2^\infty(TM) \rightarrow C_1^\infty(TM)$  of  $\hat{\nabla}$  is a tensor field of type  $(2, 1)$ .

**Exercise 6.2.** Find a proof for Proposition 6.8.

**Exercise 6.3.** Find a proof for Proposition 6.9.

**Exercise 6.4.** Let  $\mathbf{SO}(m)$  be the special orthogonal group equipped with the metric

$$\langle X, Y \rangle = \frac{1}{2} \text{trace}(X^t \cdot Y).$$

Prove that  $\langle, \rangle$  is left-invariant and that for left-invariant vector fields  $X, Y \in \mathfrak{so}(m)$  we have  $\nabla_X Y = \frac{1}{2}[X, Y]$ . Let  $A, B, C$  be elements of the Lie algebra  $\mathfrak{so}(3)$  with

$$A_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, C_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Prove that  $\{A, B, C\}$  is an orthonormal basis for  $\mathfrak{so}(3)$  and calculate  $(\nabla_A B)_e$ ,  $(\nabla_B C)_e$  and  $(\nabla_C A)_e$ .

**Exercise 6.5.** Let  $\mathbf{SL}(\mathbb{R}^2)$  be the real special linear group equipped with the metric

$$\langle X, Y \rangle_p = \text{trace}((p^{-1}X)^t \cdot (p^{-1}Y)).$$

Calculate  $(\nabla_A B)_e$ ,  $(\nabla_B C)_e$  and  $(\nabla_C A)_e$  where  $A, B, C \in \mathfrak{sl}(\mathbb{R}^2)$  are given by

$$A_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**Exercise 6.6.** Let  $(N, h)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$  and  $(M, g)$  be a submanifold with the induced metric. Prove that the second fundamental form  $B$  of  $M$  in  $N$  is symmetric and tensorial in both its arguments.

**Exercise 6.7.** Find a proof for Theorem 6.14.

**Exercise 6.8.** Find a proof for Proposition 6.15.



## Geodesics

In this chapter we introduce the notion of a geodesic on a smooth manifold as a solution to a non-linear system of ordinary differential equations. We then show that geodesics are solutions to two different variational problems. They are critical points to the so called energy functional and furthermore locally shortest paths between their endpoints.

**Definition 7.1.** Let  $M$  be a smooth manifold and  $(TM, M, \pi)$  be its tangent bundle. A **vector field  $X$  along a curve  $\gamma : I \rightarrow M$**  is a curve  $X : I \rightarrow TM$  such that  $\pi \circ X = \gamma$ . By  $C_\gamma^\infty(TM)$  we denote the set of all smooth vector fields along  $\gamma$ . For  $X, Y \in C_\gamma^\infty(TM)$  and  $f \in C^\infty(I)$  we define the operations  $\cdot$  and  $+$  by

- (i)  $(f \cdot X)(t) = f(t) \cdot X(t)$ ,
- (ii)  $(X + Y)(t) = X(t) + Y(t)$ ,

These make  $(C_\gamma^\infty(TM), +, \cdot)$  into a module over  $C^\infty(I)$  and a real vector space over the constant functions on  $M$  in particular. For a given smooth curve  $\gamma : I \rightarrow M$  in  $M$  the smooth vector field  $X : I \rightarrow TM$  with  $X : t \mapsto (\gamma(t), \dot{\gamma}(t))$  is called the **tangent field** along  $\gamma$ .

The next result gives a rule for differentiating a vector field along a given curve and shows how this is related to the Levi-Civita connection.

**Proposition 7.2.** *Let  $(M, g)$  be a smooth Riemannian manifold and  $\gamma : I \rightarrow M$  be a curve in  $M$ . Then there exists a unique operator*

$$\frac{D}{dt} : C_\gamma^\infty(TM) \rightarrow C_\gamma^\infty(TM)$$

such that for all  $\lambda, \mu \in \mathbb{R}$  and  $f \in C^\infty(I)$ ,

- (i)  $D(\lambda \cdot X + \mu \cdot Y)/dt = \lambda \cdot (DX/dt) + \mu \cdot (DY/dt)$ ,
- (ii)  $D(f \cdot Y)/dt = df/dt \cdot Y + f \cdot (DY/dt)$ , and
- (iii) for each  $t_0 \in I$  there exists an open subinterval  $J_0$  of  $I$  such that  $t_0 \in J_0$  and if  $X \in C^\infty(TM)$  is a vector field with  $X_{\gamma(t)} = Y(t)$  for all  $t \in J_0$  then

$$\left(\frac{DY}{dt}\right)(t_0) = (\nabla_{\dot{\gamma}} X)_{\gamma(t_0)}.$$

PROOF. Let us first prove the uniqueness, so for the moment we assume that such an operator exists. For a point  $t_0 \in I$  choose a chart  $(U, x)$  on  $M$  and open interval  $J_0$  such that  $t_0 \in J_0$ ,  $\gamma(J_0) \subset U$  and put  $X_i = \partial/\partial x_i \in C^\infty(TU)$ . Then a vector field  $Y$  along  $\gamma$  can be written in the form

$$Y(t) = \sum_{k=1}^m \alpha_k(t) (X_k)_{\gamma(t)}$$

for some functions  $\alpha_k \in C^\infty(J_0)$ . The second condition means that

$$(1) \quad \left(\frac{DY}{dt}\right)(t) = \sum_{k=1}^m \alpha_k(t) \left(\frac{DX_k}{dt}\right)_{\gamma(t)} + \sum_{k=1}^m \dot{\alpha}_k(t) (X_k)_{\gamma(t)}.$$

Let  $x \circ \gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$  then

$$\dot{\gamma}(t) = \sum_{k=1}^m \dot{\gamma}_k(t) (X_k)_{\gamma(t)}$$

and the third condition for  $D/dt$  imply that

$$(2) \quad \left(\frac{DX_j}{dt}\right)_{\gamma(t)} = (\nabla_{\dot{\gamma}} X_j)_{\gamma(t)} = \sum_{k=1}^m \dot{\gamma}_k(t) (\nabla_{X_k} X_j)_{\gamma(t)}.$$

Together equations (1) and (2) give

$$(3) \quad \left(\frac{DY}{dt}\right)(t) = \sum_{k=1}^m \left\{ \dot{\alpha}_k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k(\gamma(t)) \dot{\gamma}_i(t) \alpha_j(t) \right\} (X_k)_{\gamma(t)}.$$

This shows that the operator  $D/dt$  is uniquely determined.

It is easily seen that if we use equation (3) for defining an operator  $D/dt$  then it satisfies the necessary conditions of Proposition 7.2. This proves the existence of the operator  $D/dt$ .  $\square$

**Remark 7.3.** It follows from the fact that the Levi-Civita connection is tensorial in its first argument i.e.

$$\nabla_f \cdot Z X = f \cdot \nabla_Z X$$

and the equation

$$\left(\frac{DY}{dt}\right)(t_0) = (\nabla_{\dot{\gamma}} X)_{\gamma(t_0)}$$

in Proposition 7.2 that the value  $(\nabla_Z X)_p$  of  $\nabla_Z X$  at  $p$  only depends on the value of  $Z_p$  of  $Z$  at  $p$  and the values of  $Y$  along some curve  $\gamma$  satisfying  $\gamma(0) = p$  and  $\dot{\gamma}(0) = Z_p$ . This allows us to use the notation  $\nabla_{\dot{\gamma}} Y$  for  $DY/dt$ .

The Levi-Civita connection can now be used to define a parallel vector field and a geodesic on a manifold as solutions to ordinary differential equations

**Definition 7.4.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a  $C^1$ -curve. A vector field  $X$  along  $\gamma$  is said to be **parallel** along  $\gamma$  if

$$\nabla_{\dot{\gamma}} X = 0.$$

A  $C^2$ -curve  $\gamma : I \rightarrow M$  is said to be a **geodesic** if the vector field  $\dot{\gamma}$  is parallel along  $\gamma$  i.e.

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

The next result shows that for given initial values at a point  $p \in M$  we get a parallel vector field globally defined along any curve through that point.

**Theorem 7.5.** *Let  $(M, g)$  be a Riemannian manifold and  $I = (a, b)$  be an open interval on the real line  $\mathbb{R}$ . Further let  $\gamma : I \rightarrow M$  be a smooth curve,  $t_0 \in I$  and  $X_0 \in T_{\gamma(t_0)}M$ . Then there exists a unique parallel vector field  $Y$  along  $\gamma$  such that  $X_0 = Y(t_0)$ .*

**PROOF.** Without loss of generality we may assume that the image of  $\gamma$  lies in a chart  $(U, x)$ . We put  $X_i = \partial/\partial x_i$  so on the interval  $I$  the tangent field  $\dot{\gamma}$  is represented in our local coordinates by

$$\dot{\gamma}(t) = \sum_{i=1}^m \rho_i(t) (X_i)_{\gamma(t)}$$

with some functions  $\rho_i \in C^\infty(I)$ . Similarly let  $Y$  be a vector field along  $\gamma$  represented by

$$Y(t) = \sum_{j=1}^m \sigma_j(t) (X_j)_{\gamma(t)}.$$

Then

$$\begin{aligned} (\nabla_{\dot{\gamma}} Y)(t) &= \sum_{j=1}^m \{ \dot{\sigma}_j(t) (X_j)_{\gamma(t)} + \sigma_j(t) (\nabla_{\dot{\gamma}} X_j)_{\gamma(t)} \} \\ &= \sum_{k=1}^m \{ \dot{\sigma}_k(t) + \sum_{i,j=1}^m \sigma_j(t) \rho_i(t) \Gamma_{ij}^k(\gamma(t)) \} (X_k)_{\gamma(t)}. \end{aligned}$$

This implies that the vector field  $Y$  is parallel i.e.  $\nabla_{\dot{\gamma}} Y \equiv 0$  if and only if the following **linear** system of ordinary differential equations is

satisfied:

$$\dot{\sigma}_k(t) + \sum_{i,j=1}^m \sigma_j(t) \rho_i(t) \Gamma_{ij}^k(\gamma(t)) = 0$$

for all  $k = 1, \dots, m$ . It follows from classical results on ordinary differential equations that to each initial value  $\sigma(t_0) = (v_1, \dots, v_m) \in \mathbb{R}^m$  with

$$X_0 = \sum_{k=1}^m v_k (X_k)_{\gamma(t_0)}$$

there exists a unique solution  $\sigma = (\sigma_1, \dots, \sigma_m)$  to the above system. This gives us the unique parallel vector field  $Y$

$$Y(t) = \sum_{k=1}^m \sigma_k(t) (X_k)_{\gamma(t)}$$

along  $I$ . □

The following result shows that parallel vector fields are useful tools in Riemannian geometry.

**Lemma 7.6.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a smooth curve and  $X, Y$  be parallel vector fields along  $\gamma$ . Then the function  $g(X, Y) : I \rightarrow \mathbb{R}$  given by  $t \mapsto g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)})$  is constant. In particular if  $\gamma$  is a geodesic then  $g(\dot{\gamma}, \dot{\gamma})$  is constant along  $\gamma$ .*

PROOF. Using the fact that the Levi-Civita connection is metric we obtain

$$\frac{d}{dt}(g(X, Y)) = g(\nabla_{\dot{\gamma}} X, Y) + g(X, \nabla_{\dot{\gamma}} Y) = 0.$$

This proves that the function  $g(X, Y)$  is constant along  $\gamma$ . □

**Proposition 7.7.** *Let  $(M, g)$  be a Riemannian manifold,  $p \in M$  and  $\{v_1, \dots, v_m\}$  be an orthonormal basis for the tangent space  $T_p M$ . Let  $\gamma : I \rightarrow M$  be a smooth curve such that  $\gamma(0) = p$  and  $X_1, \dots, X_m$  be parallel vector fields along  $\gamma$  such that  $X_k(0) = v_k$  for  $k = 1, 2, \dots, m$ . Then the set  $\{X_1(t), \dots, X_m(t)\}$  is a orthonormal basis for the tangent space  $T_{\gamma(t)} M$  for all  $t \in I$ .*

PROOF. This is a direct consequence of Lemma 7.6. □

The important geodesic equation is in general a non-linear ordinary differential equation. For this we have the following local existence result.

**Theorem 7.8.** *Let  $(M, g)$  be a Riemannian manifold. If  $p \in M$  and  $v \in T_p M$  then there exists an open interval  $I = (-\epsilon, \epsilon)$  and a unique geodesic  $\gamma : I \rightarrow M$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .*

PROOF. Let  $(U, x)$  be a chart on  $M$  such that  $p \in U$  and put  $X_i = \partial/\partial x_i$ . For an open subinterval  $J$  of  $I$  and a  $C^2$ -curve  $\gamma : J \rightarrow U$  we put  $\gamma_i = x_i \circ \gamma : J \rightarrow \mathbb{R}$ . The curve  $x \circ \gamma : J \rightarrow \mathbb{R}^m$  is  $C^2$  so we have

$$(dx)_{\gamma(t)}(\dot{\gamma}(t)) = \sum_{i=1}^m \dot{\gamma}_i(t) e_i$$

giving

$$\dot{\gamma}(t) = \sum_{i=1}^m \dot{\gamma}_i(t) (X_i)_{\gamma(t)}.$$

By differentiation we then obtain

$$\begin{aligned} \nabla_{\dot{\gamma}} \dot{\gamma} &= \sum_{j=1}^m \nabla_{\dot{\gamma}} (\dot{\gamma}_j(t) (X_j)_{\gamma(t)}) \\ &= \sum_{j=1}^m \{ \ddot{\gamma}_j(t) (X_j)_{\gamma(t)} + \sum_{i=1}^m \dot{\gamma}_j(t) \dot{\gamma}_i(t) (\nabla_{X_i} X_j)_{\gamma(t)} \} \\ &= \sum_{k=1}^m \{ \ddot{\gamma}_k(t) + \sum_{i,j=1}^m \dot{\gamma}_j(t) \dot{\gamma}_i(t) \Gamma_{ij}^k \circ \gamma(t) \} (X_k)_{\gamma(t)}. \end{aligned}$$

Hence the curve  $\gamma$  is a geodesic if and only if

$$\ddot{\gamma}_k(t) + \sum_{i,j=1}^m \dot{\gamma}_j(t) \dot{\gamma}_i(t) \Gamma_{ij}^k(\gamma(t)) = 0$$

for all  $k = 1, \dots, m$ . It follows from classical results on ordinary differential equations that for initial values  $q_0 = x(p)$  and  $w_0 = (dx)_p(v)$  there exists an open interval  $(-\epsilon, \epsilon)$  and a unique solution  $(\gamma_1, \dots, \gamma_m)$  satisfying the initial conditions

$$(\gamma_1(0), \dots, \gamma_m(0)) = q_0 \quad \text{and} \quad (\dot{\gamma}_1(0), \dots, \dot{\gamma}_m(0)) = w_0.$$

□

Let  $E^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$  be the Euclidean space. For the trivial chart  $\text{id}_{\mathbb{R}^m} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  the metric is given by  $g_{ij} = \delta_{ij}$ , so  $\Gamma_{ij}^k = 0$  for all  $i, j, k = 1, \dots, m$ . This means that  $\gamma : I \rightarrow \mathbb{R}^m$  is a geodesic if and only if  $\ddot{\gamma}(t) = 0$  or equivalently  $\gamma(t) = t \cdot a + b$  for some  $a, b \in \mathbb{R}^m$ . This proves that the geodesics are the straight lines.

**Definition 7.9.** A geodesic  $\gamma : I \rightarrow (M, g)$  in a Riemannian manifold is said to be **maximal** if it cannot be extended to a geodesic defined on an interval  $J$  strictly containing  $I$ . The manifold  $(M, g)$  is said to be **complete** if for each point  $(p, v) \in TM$  there exists a geodesic  $\gamma : \mathbb{R} \rightarrow M$  defined on the whole of  $\mathbb{R}$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .

**Proposition 7.10.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold equipped with the induced metric  $g$ . A curve  $\gamma : I \rightarrow M$  is a geodesic in  $M$  if and only if  $(\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma})^\top = 0$ .

PROOF. The statement follows directly from the fact that  $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^\top$ .  $\square$

**Example 7.11.** Let  $E^{m+1}$  be the  $(m+1)$ -dimensional Euclidean space and  $S^m$  be the unit sphere in  $E^{m+1}$  with the induced metric. At a point  $p \in S^m$  the normal space  $N_p S^m$  of  $S^m$  in  $E^{m+1}$  is simply the line spanned by  $p$ . If  $\gamma : I \rightarrow S^m$  is a curve on the sphere, then

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma}^\top = \ddot{\gamma} - \ddot{\gamma}^\perp = \ddot{\gamma} - \langle \ddot{\gamma}, \gamma \rangle \gamma.$$

This shows that  $\gamma$  is a geodesic on the sphere  $S^m$  if and only if

$$(4) \quad \ddot{\gamma} = \langle \ddot{\gamma}, \gamma \rangle \gamma.$$

For a point  $(p, v) \in TS^m$  define the curve  $\gamma = \gamma_{(p,v)} : \mathbb{R} \rightarrow S^m$  by

$$\gamma : t \mapsto \begin{cases} p & \text{if } v = 0 \\ \cos(|v|t) \cdot p + \sin(|v|t) \cdot v/|v| & \text{if } v \neq 0. \end{cases}$$

Then one easily checks that  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = v$  and that  $\gamma$  satisfies the geodesic equation (4). This shows that the non-constant geodesics on  $S^m$  are precisely the great circles and the sphere is complete.

**Example 7.12.** Let  $\text{Sym}(\mathbb{R}^{m+1})$  be equipped with the metric

$$\langle A, B \rangle = \frac{1}{8} \text{trace}(A^t \cdot B).$$

Then we know that the map  $\phi : S^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  with

$$\phi : p \mapsto (2pp^t - e)$$

is an isometric immersion and that the image  $\phi(S^m)$  is isometric to the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ . This means that the geodesics on  $\mathbb{R}P^m$  are exactly the images of geodesics on  $S^m$ . This shows that the real projective spaces are complete.

**Definition 7.13.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a  $C^r$ -curve on  $M$ . A **variation** of  $\gamma$  is a  $C^r$ -map  $\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$  such that for all  $s \in I$ ,  $\Phi_0(s) = \Phi(0, s) = \gamma(s)$ . If the interval is compact i.e. of the form  $I = [a, b]$ , then the variation  $\Phi$  is called **proper** if for all  $t \in (-\epsilon, \epsilon)$ ,  $\Phi_t(a) = \gamma(a)$  and  $\Phi_t(b) = \gamma(b)$ .

**Definition 7.14.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a  $C^2$ -curve on  $M$ . For every compact interval  $[a, b] \subset I$  we define the energy functional  $E_{[a,b]}$  by

$$E_{[a,b]}(\gamma) = \frac{1}{2} \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

A  $C^2$ -curve  $\gamma : I \rightarrow M$  is called a **critical point** for the energy functional if every proper variation  $\Phi$  of  $\gamma|_{[a,b]}$  satisfies

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = 0.$$

We shall now prove that the geodesics can be characterized as being the critical points of the energy functional.

**Theorem 7.15.** *A  $C^2$ -curve  $\gamma$  is a critical point for the energy functional if and only if it is a geodesic.*

PROOF. For a  $C^2$ -map  $\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$ ,  $\Phi : (t, s) \mapsto \Phi(t, s)$  we define the vector fields  $X = d\Phi(\partial/\partial s)$  and  $Y = d\Phi(\partial/\partial t)$  along  $\Phi$ . The following shows that the vector fields  $X$  and  $Y$  commute:

$$\nabla_X Y - \nabla_Y X = [X, Y] = [d\Phi(\partial/\partial s), d\Phi(\partial/\partial t)] = d\Phi([\partial/\partial s, \partial/\partial t]) = 0,$$

since  $[\partial/\partial s, \partial/\partial t] = 0$ .

We now assume that  $\Phi$  is a proper variation of  $\gamma|_{[a,b]}$ . Then

$$\begin{aligned} \frac{d}{dt}(E_{[a,b]}(\Phi_t)) &= \frac{1}{2} \frac{d}{dt} \left( \int_a^b g(X, X) ds \right) \\ &= \frac{1}{2} \int_a^b \frac{d}{dt} (g(X, X)) ds \\ &= \int_a^b g(\nabla_Y X, X) ds \\ &= \int_a^b g(\nabla_X Y, X) ds \\ &= \int_a^b \left( \frac{d}{ds} (g(Y, X)) - g(Y, \nabla_X X) \right) ds \\ &= [g(Y, X)]_a^b - \int_a^b g(Y, \nabla_X X) ds. \end{aligned}$$

The variation is proper, so  $Y(a) = Y(b) = 0$ . Furthermore  $X(0, s) = \partial\Phi/\partial s(0, s) = \dot{\gamma}(s)$ , so

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = - \int_a^b g(Y(0, s), (\nabla_{\dot{\gamma}}\dot{\gamma})(s))ds.$$

The last integral vanishes for every proper variation  $\Phi$  of  $\gamma$  if and only if  $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ .  $\square$

A geodesic  $\gamma : I \rightarrow (M, g)$  is a special case of what is called a **harmonic map**  $\phi : (M, g) \rightarrow (N, h)$  between Riemannian manifolds. Other examples are conformal immersions  $\psi : (M^2, g) \rightarrow (N, h)$  which parametrize the so called minimal surfaces in  $(N, h)$ . For a reference on harmonic maps see H. Urakawa, *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs **132**, AMS(1993).

Let  $(M^m, g)$  be an  $m$ -dimensional Riemannian manifold,  $p \in M$  and

$$S_p^{m-1} = \{v \in T_pM \mid g_p(v, v) = 1\}$$

be the unit sphere in the tangent space  $T_pM$  at  $p$ . Then every point  $w \in T_pM - \{0\}$  can be written as  $w = r_w \cdot v_w$ , where  $r_w = |w|$  and  $v_w = w/|w| \in S_p^{m-1}$ . For  $v \in S_p^{m-1}$  let  $\gamma_v : (-\alpha_v, \beta_v) \rightarrow M$  be the maximal geodesic such that  $\alpha_v, \beta_v \in \mathbb{R}^+ \cup \{\infty\}$ ,  $\gamma_v(0) = p$  and  $\dot{\gamma}_v(0) = v$ . It can be shown that the real number

$$\epsilon_p = \inf\{\alpha_v, \beta_v \mid v \in S_p^{m-1}\}.$$

is positive so the open ball

$$B_{\epsilon_p}^m(0) = \{v \in T_pM \mid g_p(v, v) < \epsilon_p^2\}$$

is non-empty. The **exponential map**  $\exp_p : B_{\epsilon_p}^m(0) \rightarrow M$  at  $p$  is defined by

$$\exp_p : w \mapsto \begin{cases} p & \text{if } w = 0 \\ \gamma_{v_w}(r_w) & \text{if } w \neq 0. \end{cases}$$

Note that for  $v \in S_p^{m-1}$  the line segment  $\lambda_v : (-\epsilon_p, \epsilon_p) \rightarrow T_pM$  with  $\lambda_v : t \mapsto t \cdot v$  is mapped onto the geodesic  $\gamma_v$  i.e. locally we have  $\gamma_v = \exp_p \circ \lambda_v$ . One can prove that the map  $\exp_p$  is smooth and it follows from its definition that the differential  $d(\exp_p)_p : T_pM \rightarrow T_pM$  is the identity map for the tangent space  $T_pM$ . Then the inverse mapping theorem tells us that there exists an  $r_p \in \mathbb{R}^+$  such that if  $U_p = B_{r_p}^m(0)$  and  $V_p = \exp_p(U_p)$  then  $\exp_p|_{U_p} : U_p \rightarrow V_p$  is a diffeomorphism parametrizing the open subset  $V_p$  of  $M$ .

The next result shows that the geodesics are locally the shortest paths between their endpoints.



**Theorem 7.16.** *Let  $(M, g)$  be a Riemannian manifold,  $p \in M$  and  $\gamma : [0, \epsilon] \rightarrow M$  be a geodesic with  $\gamma(0) = p$ . Then there exists an  $\alpha \in (0, \epsilon)$  such that for each  $q \in \gamma([0, \alpha])$ ,  $\gamma$  is the shortest path from  $p$  to  $q$ .*

PROOF. Let  $p \in M$ ,  $U = B_r^m(0) \subset T_p M$  and  $V = \exp_p(U)$  be such that the restriction  $\phi = \exp_p|_U : U \rightarrow V$  of the exponential map at  $p$  is a diffeomorphism. On  $V$  we have the metric  $g$  which we pull back via  $\phi$  to obtain  $\tilde{g} = \phi^*g$  on  $U$ . This makes  $\phi : (U, \tilde{g}) \rightarrow (V, g)$  into an isometry. It then follows from the construction of the exponential map, that the geodesics in  $(U, \tilde{g})$  through the point  $0 = \phi^{-1}(p)$  are exactly the lines  $\lambda_v : t \mapsto t \cdot v$  where  $v \in T_p M$ . Now let  $q \in B_r^m(0) - \{0\}$  and  $\lambda_q : [0, 1] \rightarrow B_r^m(0)$  be the curve  $\lambda_q : t \mapsto t \cdot q$ . Further let  $\sigma : [0, 1] \rightarrow B_r^m(0)$  be any curve such that  $\sigma(0) = 0$  and  $\sigma(1) = q$ . Along  $\sigma$  we define two vector fields  $\hat{\sigma}$  and  $\dot{\sigma}_{\text{rad}}$  by  $\hat{\sigma} : t \mapsto (\sigma(t), \sigma(t))$  and

$$\dot{\sigma}_{\text{rad}} : t \mapsto (\sigma(t), \frac{\tilde{g}_{\sigma(t)}(\dot{\sigma}(t), \hat{\sigma}(t))}{\tilde{g}_{\sigma(t)}(\hat{\sigma}(t), \hat{\sigma}(t))} \cdot \sigma(t)).$$

Then it is easily checked that

$$|\dot{\sigma}_{\text{rad}}(t)| = \frac{|\tilde{g}_{\sigma(t)}(\dot{\sigma}(t), \hat{\sigma}(t))|}{|\hat{\sigma}|},$$

and

$$\frac{d}{dt}|\hat{\sigma}(t)| = \frac{d}{dt}\sqrt{\tilde{g}_{\sigma(t)}(\hat{\sigma}(t), \hat{\sigma}(t))} = \frac{\tilde{g}_{\sigma(t)}(\dot{\sigma}, \hat{\sigma})}{|\hat{\sigma}|}.$$

Combining these two relations we obtain

$$|\dot{\sigma}_{\text{rad}}(t)| = \frac{d}{dt}|\hat{\sigma}(t)|.$$

This means that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\dot{\sigma}| dt \\ &\geq \int_0^1 |\dot{\sigma}_{\text{rad}}| dt \\ &= \int_0^1 \frac{d}{dt}|\hat{\sigma}(t)| dt \\ &= |\hat{\sigma}(1)| - |\hat{\sigma}(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact  $\gamma$  is the shortest path connecting  $p$  and  $q$ .  $\square$

**Definition 7.17.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Then the **mean curvature vector field** of  $M$  in  $N$  is the smooth section  $H : M \rightarrow NM$  of the normal bundle  $NM$  given by

$$H = \frac{1}{m} \text{trace} B = \frac{1}{m} \sum_{k=1}^m B(X_k, X_k).$$

Here  $B$  is the second fundamental form of  $M$  in  $N$  and  $\{X_1, \dots, X_m\}$  is any local orthonormal frame for the tangent bundle  $TM$  of  $M$ . The submanifold  $M$  is said to be **minimal** in  $N$  if  $H \equiv 0$  and **totally geodesic** in  $N$  if  $B \equiv 0$ .

**Proposition 7.18.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold equipped with the induced metric  $g$ . Then the following conditions are equivalent:*

- (i)  $M$  is totally geodesic in  $N$
- (ii) if  $\gamma : I \rightarrow M$  is a curve, then the following conditions are equivalent
  - (a)  $\gamma : I \rightarrow M$  is a geodesic in  $M$ ,
  - (b)  $\gamma : I \rightarrow M$  is a geodesic in  $N$ .

PROOF. The result is a direct consequence of the following decomposition formula

$$\nabla_{\dot{\gamma}} \dot{\gamma} = (\nabla_{\dot{\gamma}} \dot{\gamma})^T + (\nabla_{\dot{\gamma}} \dot{\gamma})^\perp = \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} + B(\dot{\gamma}, \dot{\gamma}).$$

□

**Proposition 7.19.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$ . For a point  $(p, v)$  of the tangent bundle  $TM$  let  $\gamma_{(p,v)} : I \rightarrow N$  be the maximal geodesic in  $N$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . Then  $M$  is totally geodesic in  $(N, h)$  if and only if  $\gamma_{(p,v)}(I) \subset M$  for all  $(p, v) \in TM$ .*

PROOF. See Exercise 7.3.

□

**Proposition 7.20.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$  which is the fixpoint set of an isometry  $\phi : N \rightarrow N$ . Then  $M$  is totally geodesic in  $N$ .*

PROOF. Let  $p \in M$ ,  $v \in T_p M$  and  $\gamma : I \rightarrow N$  be the maximal geodesic with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . The map  $\phi : N \rightarrow N$  is an isometry so  $\phi \circ \gamma : I \rightarrow N$  is a geodesic. The uniqueness result of Theorem 7.8,  $\phi(\gamma(0)) = \gamma(0)$  and  $d\phi(\dot{\gamma}(0)) = \dot{\gamma}(0)$  then imply that  $\phi(\gamma) = \gamma$ . Hence the image of the geodesic  $\gamma : I \rightarrow N$  is contained in

$M$ , so following Proposition 7.19 the submanifold  $M$  is totally geodesic in  $N$ .  $\square$

**Corollary 7.21.** *If  $m < n$  then the  $m$ -dimensional sphere*

$$S^m = \{(x, 0) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} \mid |x|^2 = 1\}$$

*is totally geodesic in*

$$S^n = \{(x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} \mid |x|^2 + |y|^2 = 1\}.$$

PROOF. The statement is a direct consequence of the fact that  $S^m$  is the fixpoint set of the isometry  $\phi : S^n \rightarrow S^n$  of  $S^n$  with  $(x, y) \mapsto (x, -y)$ .  $\square$

## Exercises

**Exercise 7.1.** The result of Exercise 5.3 shows that the two dimensional hyperbolic disc  $H^2$  introduced in Example 5.5 is isometric to the upper half plane  $M = \{(x, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}^+\}$  equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{y^2} \langle X, Y \rangle_{\mathbb{R}^2}.$$

Use your local library to find all geodesics in  $(M, g)$ .

**Exercise 7.2.** Let  $n$  be a positive integer and  $\mathbf{O}(n)$  be the orthogonal group equipped with the standard left-invariant metric

$$g(A, B) = \text{trace}(A^t B).$$

Prove that a  $C^2$ -curve  $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbf{O}(n)$  is a geodesic if and only if

$$\gamma^t \cdot \ddot{\gamma} = \dot{\gamma}^t \cdot \dot{\gamma}.$$

**Exercise 7.3.** Find a proof for Proposition 7.19.

**Exercise 7.4.** For the real parameter  $\theta \in (0, \pi/2)$  define the 2-dimensional torus  $T_\theta^2$  by

$$T_\theta^2 = \{(\cos \theta e^{i\alpha}, \sin \theta e^{i\beta}) \in S^3 \mid \alpha, \beta \in \mathbb{R}\}.$$

Determine for which  $\theta \in (0, \pi/2)$  the torus  $T_\theta^2$  is a minimal submanifold of the 3-dimensional sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

**Exercise 7.5.** Show that the  $m$ -dimensional hyperbolic space

$$H^m = \{(x, 0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \mid |x| < 1\}$$

is a totally geodesic submanifold of the  $n$ -dimensional hyperbolic space  $H^n$ .

**Exercise 7.6.** Determine the totally geodesic submanifolds of the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ .

**Exercise 7.7.** Let the orthogonal group  $\mathbf{O}(n)$  be equipped with the left-invariant metric  $g(A, B) = \text{trace}(A^t B)$  and let  $K \subset \mathbf{O}(n)$  be a Lie subgroup. Prove that  $K$  is totally geodesic in  $\mathbf{O}(n)$ .

## The Curvature Tensor

In this chapter we introduce the Riemann curvature tensor as twice the skew-symmetric part of the second derivative  $\nabla^2$ . This leads to the notion of the sectional curvature which is a fundamental tool for the study of the geometry of manifolds. We prove that the spheres, Euclidean spaces and hyperbolic spaces all have constant sectional curvatures. We calculate the curvature tensor for manifolds of constant curvature and for certain Lie groups. Finally we prove the famous Gauss equation comparing the sectional curvature of a submanifold and that of its ambient space.

**Definition 8.1.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . For tensor fields  $A : C_r^\infty(TM) \rightarrow C_0^\infty(M)$  and  $B : C_r^\infty(TM) \rightarrow C_1^\infty(TM)$  we define their **covariant derivatives**  $\nabla A : C_{r+1}^\infty(TM) \rightarrow C_0^\infty(M)$  and  $\nabla B : C_{r+1}^\infty(TM) \rightarrow C_1^\infty(TM)$  by

$$\begin{aligned} \nabla A : (X, X_1, \dots, X_r) &\mapsto (\nabla_X A)(X_1, \dots, X_r) = \\ &X(A(X_1, \dots, X_r)) - \sum_{i=1}^r A(X_1, \dots, X_{i-1}, \nabla_X X_i, X_{i+1}, \dots, X_r) \end{aligned}$$

and

$$\begin{aligned} \nabla B : (X, X_1, \dots, X_r) &\mapsto (\nabla_X B)(X_1, \dots, X_r) = \\ &\nabla_X(B(X_1, \dots, X_r)) - \sum_{i=1}^r B(X_1, \dots, X_{i-1}, \nabla_X X_i, X_{i+1}, \dots, X_r). \end{aligned}$$

A tensor field  $E$  of type  $(r, 0)$  or  $(r, 1)$  is said to be **parallel** if  $\nabla E \equiv 0$ .

An example of a parallel tensor field of type  $(2, 0)$  is the Riemannian metric  $g$  of  $(M, g)$ . For this see Exercise 8.1. A vector field  $Z \in C^\infty(TM)$  defines a smooth tensor field  $\hat{Z} : C_1^\infty(TM) \rightarrow C_1^\infty(TM)$  given by

$$\hat{Z} : X \mapsto \nabla_X Z.$$

For two vector fields  $X, Y \in C^\infty(TM)$  we define the **second covariant derivative**  $\nabla^2_{X, Y} : C^\infty(TM) \rightarrow C^\infty(TM)$  by

$$\nabla^2_{X, Y} : Z \mapsto (\nabla_X \hat{Z})(Y).$$

It then follows from the definition above that

$$\nabla^2_{X, Y} Z = \nabla_X(\hat{Z}(Y)) - \hat{Z}(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z.$$

**Definition 8.2.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . Let  $R : C_3^\infty(TM) \rightarrow C_1^\infty(TM)$  be twice the skew-symmetric part of the second covariant derivative  $\nabla^2$  i.e.

$$R(X, Y)Z = \nabla^2_{X, Y} Z - \nabla^2_{Y, X} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Then  $R$  is a smooth tensor field of type  $(3, 1)$  called the **curvature tensor** of the Riemannian manifold  $(M, g)$ .

PROOF. See Exercise 8.2.  $\square$

Note that the curvature tensor  $R$  only depends on the intrinsic object  $\nabla$  and hence it is intrinsic itself. The following shows that it has many nice properties of symmetry.

**Proposition 8.3.** *Let  $(M, g)$  be a smooth Riemannian manifold. For vector fields  $X, Y, Z, W$  on  $M$  we then have*

- (i)  $R(X, Y)Z = -R(Y, X)Z,$
- (ii)  $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z),$
- (iii)  $g(R(X, Y)Z, W) + g(R(Z, X)Y, W) + g(R(Y, Z)X, W) = 0,$
- (iv)  $g(R(X, Y)Z, W) = g(R(Z, W)X, Y),$
- (v)  $6 \cdot R(X, Y)Z = R(X, Y + Z)(Y + Z) - R(X, Y - Z)(Y - Z) \\ + R(X + Z, Y)(X + Z) - R(X - Z, Y)(X - Z).$

PROOF. See Exercise 8.3.  $\square$

For a point  $p \in M$  let  $G_2(T_p M)$  denote the Grassmannian of 2-planes in  $T_p M$  i.e. the set of all 2-dimensional subspaces of  $T_p M$

$$G_2(T_p M) = \{V \subset T_p M \mid V \text{ is a 2-dimensional subspace of } T_p M\}.$$

**Definition 8.4.** For a point  $p \in M$  the function  $K_p : G_2(T_p M) \rightarrow \mathbb{R}$  given by

$$K_p : \text{span}_{\mathbb{R}}\{X, Y\} \mapsto \frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2}$$

is called the **sectional curvature** at  $p$ . Furthermore define the functions  $\delta, \Delta : M \rightarrow \mathbb{R}$  by

$$\delta : p \mapsto \min_{V \in G_2(T_p M)} K_p(V) \quad \text{and} \quad \Delta : p \mapsto \max_{V \in G_2(T_p M)} K_p(V).$$

The Riemannian manifold  $(M, g)$  is said to be

- (i) of **(strictly) positive curvature** if  $\delta(p) \geq 0$  ( $> 0$ ) for all  $p$ ,
- (ii) of **(strictly) negative curvature** if  $\Delta(p) \leq 0$  ( $< 0$ ) for all  $p$ ,
- (iii) of **constant curvature** if  $\delta = \Delta$  is constant,
- (iv) **flat** if  $\delta \equiv \Delta \equiv 0$ .

The statement of Lemma 8.5 shows that the sectional curvature just introduced is well-defined.

**Lemma 8.5.** *Let  $X, Y, Z, W \in T_p M$  be tangent vectors at  $p$  such that the two 2-dimensional subspaces  $\text{span}_{\mathbb{R}}\{X, Y\}$ ,  $\text{span}_{\mathbb{R}}\{Z, W\}$  are equal. Then*

$$\frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2} = \frac{g(R(Z, W)W, Z)}{|Z|^2|W|^2 - g(Z, W)^2}.$$

PROOF. See Exercise 8.4. □

We have the following way of expressing the curvature tensor in local coordinates.

**Proposition 8.6.** *Let  $(M, g)$  be a Riemannian manifold and let  $(U, x)$  be local coordinates on  $M$ . For  $i, j, k, l = 1, \dots, m$  put  $X_i = \partial/\partial x_i$  and  $R_{ijkl} = g(R(X_i, X_j)X_k, X_l)$ . Then*

$$R_{ijkl} = \sum_{s=1}^m g_{sl} \left( \frac{\partial \Gamma_{jk}^s}{\partial x_i} - \frac{\partial \Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^m \{ \Gamma_{jk}^r \cdot \Gamma_{ir}^s - \Gamma_{ik}^r \cdot \Gamma_{jr}^s \} \right)$$

PROOF. Using the fact that  $[X_i, X_j] = 0$  we obtain

$$\begin{aligned} R(X_i, X_j)X_k &= \nabla_{X_i} \nabla_{X_j} X_k - \nabla_{X_j} \nabla_{X_i} X_k \\ &= \nabla_{X_i} (\sum_{s=1}^m \Gamma_{jk}^s \cdot X_s) - \nabla_{X_j} (\sum_{s=1}^m \Gamma_{ik}^s \cdot X_s) \\ &= \sum_{s=1}^m \left( \frac{\partial \Gamma_{jk}^s}{\partial x_i} \cdot X_s + \sum_{r=1}^m \Gamma_{jk}^s \Gamma_{ir}^r X_r - \frac{\partial \Gamma_{ik}^s}{\partial x_j} \cdot X_s - \sum_{r=1}^m \Gamma_{ik}^s \Gamma_{jr}^r X_r \right) \\ &= \sum_{s=1}^m \left( \frac{\partial \Gamma_{jk}^s}{\partial x_i} - \frac{\partial \Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^m \{ \Gamma_{jk}^r \Gamma_{ir}^s - \Gamma_{ik}^r \Gamma_{jr}^s \} \right) X_s. \end{aligned}$$

□

For the  $m$ -dimensional vector space  $\mathbb{R}^m$  equipped with the Euclidean metric  $\langle, \rangle_{\mathbb{R}^m}$  the set  $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}$  is a global frame for the tangent bundle  $T\mathbb{R}^m$ . We have  $g_{ij} = \delta_{ij}$ , so  $\Gamma_{ij}^k \equiv 0$ . This implies that  $R \equiv 0$  so  $E^m$  is flat.

**Example 8.7.** The standard sphere  $S^m$  has constant sectional curvature  $+1$  (see Exercises 8.7 and 8.8) and the hyperbolic space  $H^m$  has constant sectional curvature  $-1$  (see Exercise 8.9).

Our next goal is Corollary 8.11 where we obtain a formula for the curvature tensor of the manifolds of constant sectional curvature  $\kappa$ . This turns out to be very useful in the study of Jacobi fields later on.

**Lemma 8.8.** *Let  $(M, g)$  be a Riemannian manifold and  $(p, Y) \in TM$ . Then the map  $\tilde{Y} : T_pM \rightarrow T_pM$  with  $\tilde{Y} : X \mapsto R(X, Y)Y$  is a symmetric endomorphism of the tangent space  $T_pM$ .*

PROOF. For  $Z \in T_pM$  we have

$$\begin{aligned} g(\tilde{Y}(X), Z) &= g(R(X, Y)Y, Z) = g(R(Y, Z)X, Y) \\ &= g(R(Z, Y)Y, X) = g(X, \tilde{Y}(Z)). \end{aligned}$$

□

Let  $(p, Y) \in T_pM$  be an element of the tangent bundle  $TM$  of  $M$  such that  $|Y| = 1$  and define

$$\mathcal{N}(Y) = \{X \in T_pM \mid g(X, Y) = 0\}.$$

The fact that  $\tilde{Y}(Y) = 0$  and Lemma 8.8 ensure the existence of an orthonormal basis of eigenvectors  $X_1, \dots, X_{m-1}$  for the restriction of the symmetric endomorphism  $\tilde{Y}$  to  $\mathcal{N}(Y)$ . The corresponding eigenvalues satisfy

$$\delta(p) \leq \lambda_1(p) \leq \dots \leq \lambda_{m-1}(p) \leq \Delta(p).$$

**Definition 8.9.** Let  $(M, g)$  be a Riemannian manifold. Then define the smooth tensor field  $R_1 : C_3^\infty(TM) \rightarrow C_1^\infty(TM)$  of type  $(3, 1)$  by

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

**Proposition 8.10.** *Let  $(M, g)$  be a smooth Riemannian manifold and  $X, Y, Z$  be vector fields on  $M$ . Then*

- (i)  $|R(X, Y)Y - \frac{\delta+\Delta}{2}R_1(X, Y)Y| \leq \frac{1}{2}(\Delta - \delta)|X||Y|^2$
- (ii)  $|R(X, Y)Z - \frac{\delta+\Delta}{2}R_1(X, Y)Z| \leq \frac{2}{3}(\Delta - \delta)|X||Y||Z|$

PROOF. Without loss of generality we can assume that  $|X| = |Y| = |Z| = 1$ . If  $X = X^\perp + X^\top$  with  $X^\perp \perp Y$  and  $X^\top$  is a multiple of  $Y$  then  $R(X, Y)Z = R(X^\perp, Y)Z$  and  $|X^\perp| \leq |X|$  so we can also assume that  $X \perp Y$ . Then  $R_1(X, Y)Y = \langle Y, Y \rangle X - \langle X, Y \rangle Y = X$ .

The first statement follows from the fact that the symmetric endomorphism of  $T_pM$  with

$$X \mapsto \left\{ R(X, Y)Y - \frac{\Delta + \delta}{2} \cdot X \right\}$$



restricted to  $\mathcal{N}(Y)$  has eigenvalues in the interval  $[\frac{\delta-\Delta}{2}, \frac{\Delta-\delta}{2}]$ .

It is easily checked that the operator  $R_1$  satisfies the conditions of Proposition 8.3 and hence  $D = R - \frac{\Delta+\delta}{2} \cdot R_1$  as well. This implies that

$$\begin{aligned} 6 \cdot D(X, Y)Z &= D(X, Y+Z)(Y+Z) - D(X, Y-Z)(Y-Z) \\ &\quad + D(X+Z, Y)(X+Z) - D(X-Z, Y)(X-Z). \end{aligned}$$

The second statement then follows from

$$\begin{aligned} 6|D(X, Y)Z| &\leq \frac{1}{2}(\Delta - \delta)\{|X|(|Y+Z|^2 + |Y-Z|^2) \\ &\quad + |Y|(|X+Z|^2 + |X-Z|^2)\} \\ &= \frac{1}{2}(\Delta - \delta)\{2|X|(|Y|^2 + |Z|^2) + 2|Y|(|X|^2 + |Z|^2)\} \\ &= 4(\Delta - \delta). \end{aligned}$$

□

As a direct consequence we have the following useful result.

**Corollary 8.11.** *Let  $(M, g)$  be a Riemannian manifold of constant curvature  $\kappa$ . Then the curvature tensor  $R$  is given by*

$$R(X, Y)Z = \kappa(\langle Y, Z \rangle X - \langle X, Z \rangle Y).$$

PROOF. This follows directly from Proposition 8.10 by using  $\Delta = \delta = \kappa$ . □

**Proposition 8.12.** *Let  $(G, \langle, \rangle)$  be a Lie group equipped with a left-invariant metric such that for all  $X \in \mathfrak{g}$  the endomorphism  $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$  is skew-symmetric with respect to  $\langle, \rangle$ . Then for any left-invariant vector fields  $X, Y, Z \in \mathfrak{g}$  the curvature tensor  $R$  is given by*

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z].$$

PROOF. See Exercise 8.6. □

We shall now define the Ricci and scalar curvatures of a Riemannian manifold. These are obtained by taking traces over the curvature tensor and play an important role in Riemannian geometry.

**Definition 8.13.** Let  $(M, g)$  be a Riemannian manifold, then

(i) the **Ricci operator**  $r : C_1^\infty(TM) \rightarrow C_1^\infty(M)$  is defined by

$$r(X) = \sum_{i=1}^m R(X, e_i)e_i,$$

(ii) the **Ricci curvature**  $Ric : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  by

$$Ric(X, Y) = \sum_{i=1}^m g(R(X, e_i)e_i, Y), \quad \text{and}$$

(iii) the **scalar curvature**  $\sigma \in C^\infty(M)$  by

$$\sigma = \sum_{j=1}^m Ric(e_j, e_j) = \sum_{j=1}^m \sum_{i=1}^m g(R(e_i, e_j)e_j, e_i).$$

Here  $\{e_1, \dots, e_m\}$  is any local orthonormal frame for the tangent bundle.

**Corollary 8.14.** *Let  $(M, g)$  be a Riemannian manifold of constant sectional curvature  $\kappa$ . Then the following holds*

$$\sigma(p) = m \cdot (m - 1) \cdot \kappa.$$

PROOF. Let  $\{e_1, \dots, e_m\}$  be an orthonormal basis, then Corollary 8.11 implies that

$$\begin{aligned} Ric_p(e_j, e_j) &= \sum_{i=1}^m g(R(e_j, e_i)e_i, e_j) \\ &= \sum_{i=1}^m g(\kappa(g(e_i, e_i)e_j - g(e_j, e_i)e_i), e_j) \\ &= \kappa \left( \sum_{i=1}^m g(e_i, e_i)g(e_j, e_j) - \sum_{i=1}^m g(e_i, e_j)g(e_i, e_j) \right) \\ &= \kappa \left( \sum_{i=1}^m 1 - \sum_{i=1}^m \delta_{ij} \right) = (m - 1) \cdot \kappa. \end{aligned}$$

To obtain the formula for the scalar curvature  $\sigma$  we only have to multiply the constant Ricci curvature  $Ric_p(e_j, e_j)$  by  $m$ .  $\square$

We complete this chapter by proving the famous Gauss equation comparing the curvature tensors of a submanifold and its ambient space in terms of the second fundamental form.

**Theorem 8.15** (The Gauss Equation). *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$  equipped with the induced metric  $g$ . Let  $X, Y, Z, W \in C^\infty(TN)$  be vector fields extending  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(TM)$ . Then*

$$\begin{aligned} \langle \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W} \rangle &= \langle R(X, Y)Z, W \rangle + \langle B(\tilde{Y}, \tilde{Z}), B(\tilde{X}, \tilde{W}) \rangle \\ &\quad - \langle B(\tilde{X}, \tilde{Z}), B(\tilde{Y}, \tilde{W}) \rangle. \end{aligned}$$

PROOF. Using the definitions of the curvature tensors  $R$ ,  $\tilde{R}$ , the Levi-Civita connection  $\tilde{\nabla}$  and the second fundamental form of  $\tilde{M}$  in  $M$  we obtain

$$\begin{aligned}
& \langle \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W} \rangle \\
&= \langle \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}, \tilde{W} \rangle \\
&= \langle (\nabla_X(\nabla_Y Z - B(Y, Z)))^T - (\nabla_Y(\nabla_X Z - B(X, Z)))^T, W \rangle \\
&\quad - \langle (\nabla_{[X, Y]} Z - B([X, Y], Z))^T, W \rangle \\
&= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W \rangle \\
&\quad - \langle \nabla_X(B(Y, Z)) - \nabla_Y(B(X, Z)), W \rangle \\
&= \langle R(X, Y)Z, W \rangle + \langle B(Y, Z), B(X, W) \rangle - \langle B(X, Z), B(Y, W) \rangle.
\end{aligned}$$

□

As a direct consequence of the Gauss equation we have the following useful formula.

**Corollary 8.16.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a totally geodesic submanifold of  $N$ . Let  $X, Y, Z, W \in C^\infty(TN)$  be vector fields extending  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(T\tilde{M})$ . Then*

$$\langle \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W} \rangle = \langle R(X, Y)Z, W \rangle.$$

## Exercises

**Exercise 8.1.** Let  $(M, g)$  be a Riemannian manifold. Prove that the tensor field  $g$  of type  $(2, 0)$  is parallel with respect to the Levi-Civita connection.

**Exercise 8.2.** Let  $(M, g)$  be a Riemannian manifold. Prove that  $R$  is a smooth tensor field of type  $(3, 1)$ .

**Exercise 8.3.** Find a proof for Proposition 8.3.

**Exercise 8.4.** Find a proof for Lemma 8.5.

**Exercise 8.5.** Let  $\mathbb{R}^m$  be equipped with the standard Euclidean metric and  $\mathbb{C}^m$  with the Euclidean metric  $g$  given by

$$g(z, w) = \sum_{k=1}^m \operatorname{Re}(z_k \bar{w}_k).$$

Let  $T^m$  be the  $m$ -dimensional torus  $\{z \in \mathbb{C}^m \mid |z_1| = \dots = |z_m| = 1\}$  in  $\mathbb{C}^m$  with the induced metric  $\tilde{g}$ . Find an isometric immersion  $\phi : \mathbb{R}^m \rightarrow T^m$ , determine all geodesics on  $(T^m, g)$  and prove that  $(T^m, g)$  is flat.

**Exercise 8.6.** Find a proof for Proposition 8.12.

**Exercise 8.7.** Let the Lie group  $S^3 \cong \mathbf{SU}(2)$  be equipped with the metric  $\langle X, Y \rangle = \frac{1}{2} \operatorname{Re}\{\operatorname{trace}(\bar{X}^t \cdot Y)\}$ .

- (i) Find an orthonormal basis for  $T_e \mathbf{SU}(2)$ .
- (ii) Show that  $(\mathbf{SU}(2), g)$  has constant sectional curvature  $+1$ .

**Exercise 8.8.** Let  $S^m$  be the unit sphere in  $\mathbb{R}^{m+1}$  equipped with the standard Euclidean metric  $\langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}}$ . Use the results of Corollaries 7.21, 8.16 and Exercise 8.7 to prove that  $(S^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}})$  has constant sectional curvature  $+1$ .

**Exercise 8.9.** Let  $H^m = (\mathbb{R}^+ \times \mathbb{R}^{m-1}, \frac{1}{x_1^2} \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$  be the  $m$ -dimensional hyperbolic space. On  $H^m$  we define the operation  $*$  by  $(\alpha, x) * (\beta, y) = (\alpha \cdot \beta, \alpha \cdot y + x)$ . For  $k = 1, \dots, m$  define the vector field  $X_k \in C^\infty(TH^m)$  by  $(X_k)_x = x_1 \cdot \frac{\partial}{\partial x_k}$ . Prove that,

- (i)  $(H^m, *)$  is a Lie group,
- (ii) the vector fields  $X_1, \dots, X_m$  are left-invariant,
- (iii) the metric  $g$  is left-invariant,
- (iv)  $(H^m, g)$  has constant curvature  $-1$ .

## Curvature and Local Geometry

This chapter is devoted to the study of the local geometry of Riemannian manifolds and how this is controlled by the curvature tensor. For this we introduce the notion of a Jacobi field which is a useful tool in differential geometry. With this in hand we yield a fundamental comparison result describing the curvature dependence of local distances.

Let  $(M, g)$  be a smooth Riemannian manifold. By a smooth **1-parameter family of geodesics** we mean a  $C^\infty$ -map

$$\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$$

such that the curve  $\gamma_t : I \rightarrow M$  given by  $\gamma_t : s \mapsto \Phi(t, s)$  is a geodesic for all  $t \in (-\epsilon, \epsilon)$ . The variable  $t \in (-\epsilon, \epsilon)$  is called the **family parameter** of  $\Phi$ .

**Proposition 9.1.** *Let  $(M, g)$  be a Riemannian manifold and  $\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$  be a 1-parameter family of geodesics. Then for each  $t \in (-\epsilon, \epsilon)$  the vector field  $J_t : I \rightarrow C^\infty(TM)$  along  $\gamma_t$  given by*

$$J_t(s) = \frac{\partial \Phi}{\partial t}(t, s)$$

*satisfies the second order ordinary differential equation*

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t) \dot{\gamma}_t = 0.$$

**PROOF.** Along  $\Phi$  we put  $X(t, s) = \partial\Phi/\partial s$  and  $J(t, s) = \partial\Phi/\partial t$ . The fact that  $[\partial/\partial t, \partial/\partial s] = 0$  implies that

$$[J, X] = [d\Phi(\partial/\partial t), d\Phi(\partial/\partial s)] = d\Phi([\partial/\partial t, \partial/\partial s]) = 0.$$

Since  $\Phi$  is a family of geodesics we have  $\nabla_X X = 0$  and the definition of the curvature tensor then gives

$$\begin{aligned} R(J, X)X &= \nabla_J \nabla_X X - \nabla_X \nabla_J X - \nabla_{[J, X]} X \\ &= -\nabla_X \nabla_J X \\ &= -\nabla_X \nabla_X J. \end{aligned}$$

Hence for each  $t \in (-\epsilon, \epsilon)$  we have

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t) \dot{\gamma}_t = 0.$$

□

This leads to the following definition.

**Definition 9.2.** Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic and  $X = \dot{\gamma}$ . A  $C^2$  vector field  $J$  along  $\gamma$  is called a **Jacobi field** if

$$(5) \quad \nabla_X \nabla_X J + R(J, X)X = 0$$

along  $\gamma$ . We denote the space of all Jacobi fields along  $\gamma$  by  $\mathcal{J}_\gamma(TM)$ .

We shall now give an example of a 1-parameter family of geodesics in the  $(m + 1)$ -dimensional Euclidean space  $E^{m+1}$ .

**Example 9.3.** Let  $c, n : \mathbb{R} \rightarrow E^{m+1}$  be smooth curves such that the image  $n(\mathbb{R})$  of  $n$  is contained in the unit sphere  $S^m$ . If we define a map  $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow E^{m+1}$  by

$$\Phi : (t, s) \mapsto c(t) + s \cdot n(t)$$

then for each  $t \in \mathbb{R}$  the curve  $\gamma_t : s \mapsto \Phi(t, s)$  is a straight line and hence a geodesic in  $E^{m+1}$ . By differentiating with respect to the family parameter  $t$  we yield the Jacobi field  $J \in \mathcal{J}_{\gamma_0}(TE^{m+1})$  along  $\gamma_0$  with

$$J(s) = \frac{d}{dt} \Phi(t, s)|_{t=0} = \dot{c}(0) + s \cdot \dot{n}(0).$$

The Jacobi equation (5) on a Riemannian manifold is linear in  $J$ . This means that the space of Jacobi fields  $\mathcal{J}_\gamma(TM)$  along  $\gamma$  is a vector space. We are now interested in determining the dimension of this space

**Proposition 9.4.** *Let  $\gamma : I \rightarrow M$  be a geodesic,  $0 \in I$ ,  $p = \gamma(0)$  and  $X = \dot{\gamma}$  along  $\gamma$ . If  $v, w \in T_p M$  are two tangent vectors at  $p$  then there exists a unique Jacobi field  $J$  along  $\gamma$ , such that  $J_p = v$  and  $(\nabla_X J)_p = w$ .*

**PROOF.** Let  $\{X_1, \dots, X_m\}$  be an orthonormal frame of parallel vector fields along  $\gamma$ . If  $J$  is a vector field along  $\gamma$ , then

$$J = \sum_{i=1}^m a_i X_i$$

where  $a_i = \langle J, X_i \rangle$  are smooth functions on  $I$ . The vector fields  $X_1, \dots, X_m$  are parallel so

$$\nabla_X J = \sum_{i=1}^m \dot{a}_i X_i \quad \text{and} \quad \nabla_X \nabla_X J = \sum_{i=1}^m \ddot{a}_i X_i.$$

For the curvature tensor we have

$$R(X_i, X)X = \sum_{k=1}^m b_i^k X_k,$$

where  $b_i^k = \langle R(X_i, X)X, X_k \rangle$  are smooth functions on  $I$  depending on the geometry of  $(M, g)$ . This means that  $R(J, X)X$  is given by

$$R(J, X)X = \sum_{i,k=1}^m a_i b_i^k X_k.$$

and that  $J$  is a Jacobi field if and only if

$$\sum_{i=1}^m (\ddot{a}_i + \sum_{k=1}^m a_k b_k^i) X_i = 0.$$

This is equivalent to the second order system

$$\ddot{a}_i + \sum_{k=1}^m a_k b_k^i = 0 \quad \text{for all } i = 1, 2, \dots, m$$

of linear ordinary differential equations in  $a = (a_1, \dots, a_m)$ . A global solution will always exist and is uniquely determined by  $a(0)$  and  $\dot{a}(0)$ . This implies that  $J$  exists globally and is uniquely determined by the initial conditions

$$J(0) = v \quad \text{and} \quad (\nabla_X J)(0) = w.$$

□

The last result has the following interesting consequence.

**Corollary 9.5.** *Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold and  $\gamma : I \rightarrow M$  be a geodesic in  $M$ . Then the vector space  $\mathcal{J}_\gamma(TM)$  of all Jacobi fields along  $\gamma$  has the dimension  $2m$ .*

The following Lemma shows that when proving results about Jacobi fields along a geodesic  $\gamma$  we can always assume, without loss of generality, that  $|\dot{\gamma}| = 1$ .

**Lemma 9.6.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic and  $J$  be a Jacobi field along  $\gamma$ . If  $\lambda \in \mathbb{R}^*$  and  $\sigma : \lambda I \rightarrow I$  is given by  $\sigma : t \mapsto t/\lambda$ , then  $\gamma \circ \sigma : \lambda I \rightarrow M$  is a geodesic and  $J \circ \sigma$  is a Jacobi field along  $\gamma \circ \sigma$ .*

PROOF. See Exercise 9.1.  $\square$

Next we determine the Jacobi fields which are tangential to a given geodesic.

**Proposition 9.7.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic with  $|\dot{\gamma}| = 1$  and  $J$  be a Jacobi field along  $\gamma$ . Let  $J^\top$  be the tangential part of  $J$  given by  $J^\top = \langle J, \dot{\gamma} \rangle \dot{\gamma}$  and  $J^\perp = J - J^\top$  be its normal part. Then  $J^\top$  and  $J^\perp$  are Jacobi fields along  $\gamma$  and there exist  $a, b \in \mathbb{R}$  such that  $J^\top(s) = (as + b)\dot{\gamma}(s)$  for all  $s \in I$ .*

PROOF. We now have

$$\begin{aligned} \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J^\top + R(J^\top, \dot{\gamma})\dot{\gamma} &= \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} (\langle J, \dot{\gamma} \rangle \dot{\gamma}) + R(\langle J, \dot{\gamma} \rangle \dot{\gamma}, \dot{\gamma})\dot{\gamma} \\ &= \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J, \dot{\gamma} \rangle \dot{\gamma} \\ &= -\langle R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma} \rangle \dot{\gamma} \\ &= 0. \end{aligned}$$

This shows that the tangential part  $J^\top$  of  $J$  is a Jacobi field. The fact that  $\mathcal{J}_\gamma(TM)$  is a vector space implies that the normal part  $J^\perp = J - J^\top$  of  $J$  also is a Jacobi field.

By differentiating  $\langle J, \dot{\gamma} \rangle$  twice along  $\gamma$  we obtain

$$\frac{d^2}{ds^2} \langle J, \dot{\gamma} \rangle = \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J, \dot{\gamma} \rangle = -\langle R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma} \rangle = 0$$

so  $\langle J, \dot{\gamma} \rangle(s) = (as + b)$  for some  $a, b \in \mathbb{R}$ .  $\square$

**Corollary 9.8.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic and  $J$  be a Jacobi field along  $\gamma$ . If  $g(J(t_0), \dot{\gamma}(t_0)) = 0$  and  $g((\nabla_{\dot{\gamma}} J)(t_0), \dot{\gamma}(t_0)) = 0$  for some  $t_0 \in I$ , then  $g(J(t), \dot{\gamma}(t)) = 0$  for all  $t \in I$ .*

PROOF. This is a direct consequence of the fact that the function  $g(J, \dot{\gamma})$  satisfies the second order ODE  $\ddot{f} = 0$  and the initial values  $f(0) = 0$  and  $\dot{f}(0) = 0$ .  $\square$

Our next aim is to show that if the Riemannian manifold  $(M, g)$  has constant sectional curvature then we can solve the Jacobi equation

$$\nabla_X \nabla_X J + R(J, X)X = 0$$

along any given geodesic  $\gamma : I \rightarrow M$ . For this we introduce the following notation. For a real number  $\kappa \in \mathbb{R}$  we define the  $c_\kappa, s_\kappa : \mathbb{R} \rightarrow \mathbb{R}$



by

$$c_\kappa(s) = \begin{cases} \cosh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0, \\ 1 & \text{if } \kappa = 0, \\ \cos(\sqrt{\kappa}s) & \text{if } \kappa > 0. \end{cases}$$

and

$$s_\kappa(s) = \begin{cases} \sinh(\sqrt{|\kappa|}s)/\sqrt{|\kappa|} & \text{if } \kappa < 0, \\ s & \text{if } \kappa = 0, \\ \sin(\sqrt{\kappa}s)/\sqrt{\kappa} & \text{if } \kappa > 0. \end{cases}$$

It is a well known fact that the unique solution to the initial value problem

$$\ddot{f} + \kappa \cdot f = 0, \quad f(0) = a \quad \text{and} \quad \dot{f}(0) = b$$

is the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $f(s) = ac_\kappa(s) + bs_\kappa(s)$ .

**Example 9.9.** Let  $\mathbb{C}$  be the complex plane with the standard Euclidean metric  $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$  of constant sectional curvature  $\kappa = 0$ . The rotations about the origin produce a 1-parameter family of geodesics  $\Phi_t : s \mapsto se^{it}$ . Along the geodesic  $\gamma_0 : s \mapsto s$  we get the Jacobi field  $J_0(s) = \partial\Phi_t/\partial t(0, s) = is$  with  $|J_0(s)| = |s| = |s_\kappa(s)|$ .

**Example 9.10.** Let  $S^2$  be the unit sphere in the standard Euclidean 3-space  $\mathbb{C} \times \mathbb{R}$  with the induced metric of constant sectional curvature  $\kappa = +1$ . Rotations about the  $\mathbb{R}$ -axis produce a 1-parameter family of geodesics  $\Phi_t : s \mapsto (\sin(s)e^{it}, \cos(s))$ . Along the geodesic  $\gamma_0 : s \mapsto (\sin(s), \cos(s))$  we get the Jacobi field  $J_0(s) = \partial\Phi_t/\partial t(0, s) = (i\sin(s), 0)$  with  $|J_0(s)|^2 = \sin^2(s) = |s_\kappa(s)|^2$ .

**Example 9.11.** Let  $B_1^2(0)$  be the open unit disk in the complex plane with the hyperbolic metric  $4/(1-|z|^2)^2 \langle \cdot, \cdot \rangle_{\mathbb{R}^2}$  of constant sectional curvature  $\kappa = -1$ . Rotations about the origin produce a 1-parameter family of geodesics  $\Phi_t : s \mapsto \tanh(s/2)e^{it}$ . Along the geodesic  $\gamma_0 : s \mapsto \tanh(s/2)$  we get the Jacobi field  $J_0(s) = i \cdot \tanh(s/2)$  with

$$|J_0(s)|^2 = \frac{4 \cdot \tanh^2(s/2)}{1 - \tanh^2(s/2)} = \sinh^2(s) = |s_\kappa(s)|^2.$$

Let  $(M, g)$  be a Riemannian manifold of constant sectional curvature  $\kappa$  and  $\gamma : I \rightarrow M$  be a geodesic with  $|X| = 1$  where  $X = \dot{\gamma}$ . Further let  $P_1, P_2, \dots, P_{m-1}$  be parallel vector fields along  $\gamma$  such that  $g(P_i, P_j) = \delta_{ij}$  and  $g(P_i, X) = 0$ . Any vector field  $J$  along  $\gamma$  may now be written as

$$J(s) = \sum_{i=1}^{m-1} f_i(s)P_i(s) + f_m(s)X(s).$$

This means that  $J$  is a Jacobi field if and only if

$$\begin{aligned}
\sum_{i=1}^{m-1} \ddot{f}_i(s)P_i(s) + \ddot{f}_m(s)X(s) &= \nabla_X \nabla_X J \\
&= -R(J, X)X \\
&= -R(J^\perp, X)X \\
&= -\kappa(g(X, X)J^\perp - g(J^\perp, X)X) \\
&= -\kappa J^\perp \\
&= -\kappa \sum_{i=1}^{m-1} f_i(s)P_i(s).
\end{aligned}$$

This is equivalent to the following system of ordinary differential equations

$$(6) \quad \ddot{f}_m(s) = 0 \quad \text{and} \quad \ddot{f}_i(s) + \kappa f_i(s) = 0 \quad \text{for all } i = 1, 2, \dots, m-1.$$

It is clear that for the initial values

$$\begin{aligned}
J(s_0) &= \sum_{i=1}^{m-1} v_i P_i(s_0) + v_m X(s_0), \\
(\nabla_X J)(s_0) &= \sum_{i=1}^{m-1} w_i P_i(s_0) + w_m X(s_0)
\end{aligned}$$

or equivalently

$$f_i(s_0) = v_i \quad \text{and} \quad \dot{f}_i(s_0) = w_i \quad \text{for all } i = 1, 2, \dots, m$$

we have a unique and explicit solution to the system (6) on the whole of  $I$ .

In the next example we give a complete description of the Jacobi fields along a geodesic on the 2-dimensional sphere.

**Example 9.12.** Let  $S^2$  be the unit sphere in the standard Euclidean 3-space  $\mathbb{C} \times \mathbb{R}$  with the induced metric of constant curvature  $\kappa = +1$  and  $\gamma : \mathbb{R} \rightarrow S^2$  be the geodesic given by  $\gamma : s \mapsto (e^{is}, 0)$ . Then  $\dot{\gamma}(s) = (ie^{is}, 0)$  so it follows from Proposition (9.7) that all Jacobi fields tangential to  $\gamma$  are given by

$$J_{(a,b)}^T(s) = (as + b)(ie^{is}, 0) \quad \text{for some } a, b \in \mathbb{R}.$$

The vector field  $P : \mathbb{R} \rightarrow TS^2$  given by  $s \mapsto ((e^{is}, 0), (0, 1))$  satisfies  $\langle P, \dot{\gamma} \rangle = 0$  and  $|P| = 1$ . The sphere  $S^2$  is 2-dimensional and  $\dot{\gamma}$  is parallel along  $\gamma$  so  $P$  must be parallel. This implies that all the Jacobi fields orthogonal to  $\dot{\gamma}$  are given by

$$J_{(a,b)}^N(s) = (0, a \cos s + b \sin s) \quad \text{for some } a, b \in \mathbb{R}.$$

In more general situations, where we do not have constant curvature the exponential map can be used to produce Jacobi fields as follows. Let  $(M, g)$  be a complete Riemannian manifold,  $p \in M$  and  $v, w \in T_p M$ . Then  $s \mapsto s(v + tw)$  defines a 1-parameter family of lines in the tangent space  $T_p M$  which all pass through the origin  $0 \in T_p M$ . Remember that the exponential map

$$(\exp)_p|_{B_{\varepsilon_p(0)}^m} : B_{\varepsilon_p(0)}^m \rightarrow \exp(B_{\varepsilon_p(0)}^m)$$

maps lines in  $T_p M$  through the origin onto geodesics on  $M$ . Hence the map

$$\Phi_t : s \mapsto (\exp)_p(s(v + tw))$$

is a 1-parameter family of geodesics through  $p \in M$ , as long as  $s(v + tw)$  is an element of  $B_{\varepsilon_p(0)}^m$ . This means that

$$J : s \mapsto (\partial\Phi_t/\partial t)(0, s)$$

is a Jacobi field along the geodesic  $\gamma : s \mapsto \Phi_0(s)$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . It is easily verified that  $J$  satisfies the initial conditions

$$J(0) = 0 \quad \text{and} \quad (\nabla_X J)(0) = w.$$

The following technical result is needed for the proof of the main theorem at the end of this chapter.

**Lemma 9.13.** *Let  $(M, g)$  be a Riemannian manifold with sectional curvature uniformly bounded above by  $\Delta$  and  $\gamma : [0, \alpha] \rightarrow M$  be a geodesic on  $M$  with  $|X| = 1$  where  $X = \dot{\gamma}$ . Further let  $J : [0, \alpha] \rightarrow TM$  be a Jacobi field along  $\gamma$  such that  $g(J, X) = 0$  and  $|J| \neq 0$  on  $(0, \alpha)$ . Then*

- (i)  $d^2(|J|)/ds^2 + \Delta \cdot |J| \geq 0$ ,
- (ii) if  $f : [0, \alpha] \rightarrow \mathbb{R}$  is a  $C^2$ -function such that
  - (a)  $\ddot{f} + \Delta \cdot f = 0$  and  $f > 0$  on  $(0, \alpha)$ ,
  - (b)  $f(0) = |J(0)|$ , and
  - (c)  $\dot{f}(0) = |\nabla_X J(0)|$ ,
 then  $f(s) \leq |J(s)|$  on  $(0, \alpha)$ ,
- (iii) if  $J(0) = 0$ , then  $|\nabla_X J(0)| \cdot s_\Delta(s) \leq |J(s)|$  for all  $s \in (0, \alpha)$ .

PROOF. (i) Using the facts that  $|X| = 1$  and  $\langle X, J \rangle = 0$  we obtain

$$\begin{aligned} \frac{d^2}{ds^2}(|J|) &= \frac{d^2}{ds^2} \sqrt{\langle J, J \rangle} = \frac{d}{ds} \left( \frac{\langle \nabla_X J, J \rangle}{|J|} \right) \\ &= \frac{\langle \nabla_X \nabla_X J, J \rangle}{|J|} + \frac{|\nabla_X J|^2 |J|^2 - \langle \nabla_X J, J \rangle^2}{|J|^3} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\langle \nabla_X \nabla_X J, J \rangle}{|J|} \\
&= -\frac{\langle R(J, X)X, J \rangle}{|J|} \\
&\geq -\Delta \cdot |J|.
\end{aligned}$$

(ii) Define the function  $h : [0, \alpha) \rightarrow \mathbb{R}$  by

$$h(s) = \begin{cases} \frac{|J(s)|}{f(s)} & \text{if } s \in (0, \alpha), \\ \lim_{s \rightarrow 0} \frac{|J(s)|}{f(s)} = 1 & \text{if } s = 0. \end{cases}$$

Then

$$\begin{aligned}
\dot{h}(s) &= \frac{1}{f^2(s)} \left( \frac{d}{ds} (|J(s)|) f(s) - |J(s)| \dot{f}(s) \right) \\
&= \frac{1}{f^2(s)} \int_0^s \left( \frac{d^2}{dt^2} (|J(t)|) f(t) - |J(t)| \ddot{f}(t) \right) dt \\
&= \frac{1}{f^2(s)} \int_0^s f(t) \left( \frac{d^2}{dt^2} (|J(t)|) + \Delta \cdot |J(t)| \right) dt \\
&\geq 0.
\end{aligned}$$

This implies that  $\dot{h}(s) \geq 0$  so  $f(s) \leq |J(s)|$  for all  $s \in (0, \alpha)$ .

(iii) The function  $f(s) = |(\nabla_X J)(0)| \cdot s_\Delta(s)$  satisfies the differential equation

$$\ddot{f}(s) + \Delta f(s) = 0$$

and the initial conditions  $f(0) = |J(0)| = 0$ ,  $\dot{f}(0) = |(\nabla_X J)(0)|$  so it follows from (ii) that  $|(\nabla_X J)(0)| \cdot s_\Delta(s) = f(s) \leq |J(s)|$ .  $\square$

Let  $(M, g)$  be a Riemannian manifold of sectional curvature which is uniformly bounded above, i.e. there exists a  $\Delta \in \mathbb{R}$  such that  $K_p(V) \leq \Delta$  for all  $V \in G_2(T_p M)$  and  $p \in M$ . Let  $(M_\Delta, g_\Delta)$  be another Riemannian manifold which is complete and of constant sectional curvature  $K \equiv \Delta$ . Let  $p \in M$ ,  $p_\Delta \in M_\Delta$  and identify  $T_p M \cong \mathbb{R}^m \cong T_{p_\Delta} M_\Delta$ .

Let  $U$  be an open neighbourhood of  $\mathbb{R}^m$  around 0 such that the exponential maps  $(\exp)_p$  and  $(\exp)_{p_\Delta}$  are diffeomorphisms from  $U$  onto their images  $(\exp)_p(U)$  and  $(\exp)_{p_\Delta}(U)$ , respectively. Let  $(r, p, q)$  be a geodesic triangle i.e. a triangle with sides which are shortest paths between their endpoints. Furthermore let  $c : [a, b] \rightarrow M$  be the side connecting  $r$  and  $q$  and  $v : [a, b] \rightarrow T_p M$  be the curve defined by  $c(t) = (\exp)_p(v(t))$ . Put  $c_\Delta(t) = (\exp)_{p_\Delta}(v(t))$  for  $t \in [a, b]$  and then it directly follows that  $c(a) = r$  and  $c(b) = q$ . Finally put  $r_\Delta = c_\Delta(a)$  and  $q_\Delta = c_\Delta(b)$ .

**Theorem 9.14.** *For the above situation the following inequality for the distance function  $d$  is satisfied*

$$d(q_\Delta, r_\Delta) \leq d(q, r).$$

PROOF. Define a 1-parameter family  $s \mapsto s \cdot v(t)$  of straight lines in  $T_p M$  through  $p$ . Then  $\Phi_t : s \mapsto (\exp)_p(s \cdot v(t))$  and  $\Phi_t^\Delta : s \mapsto (\exp)_{p_\Delta}(s \cdot v(t))$  are 1-parameter families of geodesics through  $p \in M$ , and  $p_\Delta \in M_\Delta$ , respectively. Hence  $J_t = \partial\Phi_t/\partial t$  and  $J_t^\Delta = \partial\Phi_t^\Delta/\partial t$  are Jacobi fields satisfying the initial conditions  $J_t(0) = J_t^\Delta(0) = 0$  and  $(\nabla_X J_t)(0) = (\nabla_X J_t^\Delta)(0) = \dot{v}(t)$ . Using Lemma 9.13 we now obtain

$$\begin{aligned} |\dot{c}_\Delta(t)| &= |J_t^\Delta(1)| \\ &= |(\nabla_X J_t^\Delta)(0)| \cdot s_\Delta(1) \\ &= |(\nabla_X J_t)(0)| \cdot s_\Delta(1) \leq |J_t(1)| \\ &= |\dot{c}(t)| \end{aligned}$$

The curve  $c$  is the shortest path between  $r$  and  $q$  so we have

$$d(r_\Delta, q_\Delta) \leq L(c_\Delta) \leq L(c) = d(r, q).$$

□

We now add the assumption that the sectional curvature of the manifold  $(M, g)$  is uniformly bounded below i.e. there exists a  $\delta \in \mathbb{R}$  such that  $\delta \leq K_p(V)$  for all  $V \in G_2(T_p M)$  and  $p \in M$ . Let  $(M_\delta, g_\delta)$  be a complete Riemannian manifold of constant sectional curvature  $\delta$ . Let  $p \in M$  and  $p_\delta \in M_\delta$  and identify  $T_p M \cong \mathbb{R}^m \cong T_{p_\delta} M_\delta$ . Then a similar construction as above gives two pairs of points  $q, r \in M$  and  $q_\delta, r_\delta \in M_\delta$  and shows that

$$d(q, r) \leq d(q_\delta, r_\delta).$$

Combining these two results we obtain **locally**

$$d(q_\Delta, r_\Delta) \leq d(q, r) \leq d(q_\delta, r_\delta).$$

## Exercises

**Exercise 9.1.** Find a proof for Lemma 9.6.

**Exercise 9.2.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a geodesic such that  $X = \dot{\gamma} \neq 0$ . Further let  $J$  be a non-vanishing Jacobi field along  $\gamma$  with  $g(X, J) = 0$ . Prove that if  $g(J, J)$  is constant along  $\gamma$  then  $(M, g)$  does not have strictly negative curvature.