

Chapter 8

The Dual Space, Duality

8.1 The Dual Space E^* and Linear Forms

In Section 1.7 we defined linear forms, the dual space $E^* = \text{Hom}(E, K)$ of a vector space E , and showed the existence of dual bases for vector spaces of finite dimension.

In this chapter, we take a deeper look at the connection between a space E and its dual space E^* .

As we will see shortly, every linear map $f: E \rightarrow F$ gives rise to a linear map $f^\top: F^* \rightarrow E^*$, and it turns out that in a suitable basis, the matrix of f^\top is the *transpose* of the matrix of f .

Thus, the notion of dual space provides a conceptual explanation of the phenomena associated with transposition.

But it does more, because it allows us to view subspaces as solutions of sets of linear equations and vice-versa.

Consider the following set of two “linear equations” in \mathbb{R}^3 ,

$$\begin{aligned}x - y + z &= 0 \\x - y - z &= 0,\end{aligned}$$

and let us find out what is their set V of common solutions $(x, y, z) \in \mathbb{R}^3$.

By subtracting the second equation from the first, we get $2z = 0$, and by adding the two equations, we find that $2(x - y) = 0$, so the set V of solutions is given by

$$\begin{aligned}y &= x \\z &= 0.\end{aligned}$$

This is a one dimensional subspace of \mathbb{R}^3 . Geometrically, this is the line of equation $y = x$ in the plane $z = 0$.

Now, why did we say that the above equations are linear?

This is because, as functions of (x, y, z) , both maps $f_1: (x, y, z) \mapsto x - y + z$ and $f_2: (x, y, z) \mapsto x - y - z$ are linear.

The set of all such linear functions from \mathbb{R}^3 to \mathbb{R} is a vector space; we used this fact to form linear combinations of the “equations” f_1 and f_2 .

Observe that the dimension of the subspace V is 1.

The ambient space has dimension $n = 3$ and there are two “independent” equations f_1, f_2 , so it appears that the dimension $\dim(V)$ of the subspace V defined by m independent equations is

$$\dim(V) = n - m,$$

which is indeed a general fact (proved in Theorem 8.1).

More generally, in \mathbb{R}^n , a linear equation is determined by an n -tuple $(a_1, \dots, a_n) \in \mathbb{R}^n$, and the solutions of this linear equation are given by the n -tuples $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that

$$a_1x_1 + \cdots + a_nx_n = 0;$$

these solutions constitute the kernel of the linear map $(x_1, \dots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n$.

The above considerations assume that we are working in the canonical basis (e_1, \dots, e_n) of \mathbb{R}^n , but we can define “linear equations” independently of bases and in any dimension, by viewing them as elements of the vector space $\text{Hom}(E, K)$ of linear maps from E to the field K .

Definition 8.1. Given a vector space E , the vector space $\text{Hom}(E, K)$ of linear maps from E to K is called the *dual space (or dual)* of E . The space $\text{Hom}(E, K)$ is also denoted by E^* , and the linear maps in E^* are called *the linear forms*, or *covectors*. The dual space E^{**} of the space E^* is called the *bidual* of E .

As a matter of notation, linear forms $f: E \rightarrow K$ will also be denoted by starred symbol, such as u^* , x^* , etc.

8.2 Pairing and Duality Between E and E^*

Given a linear form $u^* \in E^*$ and a vector $v \in E$, the result $u^*(v)$ of applying u^* to v is also denoted by $\langle u^*, v \rangle$.

This defines a binary operation $\langle -, - \rangle: E^* \times E \rightarrow K$ satisfying the following properties:

$$\begin{aligned}\langle u_1^* + u_2^*, v \rangle &= \langle u_1^*, v \rangle + \langle u_2^*, v \rangle \\ \langle u^*, v_1 + v_2 \rangle &= \langle u^*, v_1 \rangle + \langle u^*, v_2 \rangle \\ \langle \lambda u^*, v \rangle &= \lambda \langle u^*, v \rangle \\ \langle u^*, \lambda v \rangle &= \lambda \langle u^*, v \rangle.\end{aligned}$$

The above identities mean that $\langle -, - \rangle$ is a *bilinear map*, since it is linear in each argument.

It is often called the *canonical pairing* between E^* and E .

In view of the above identities, given any fixed vector $v \in E$, the map $\text{eval}_v: E^* \rightarrow K$ (*evaluation at v*) defined such that

$$\text{eval}_v(u^*) = \langle u^*, v \rangle = u^*(v) \quad \text{for every } u^* \in E^*$$

is a linear map from E^* to K , that is, eval_v is a linear form in E^{**} .

Again from the above identities, the map $\text{eval}_E: E \rightarrow E^{**}$, defined such that

$$\text{eval}_E(v) = \text{eval}_v \quad \text{for every } v \in E,$$

is a linear map.

We shall see that it is injective, and that it is an isomorphism when E has finite dimension.

We now formalize the notion of the set V^0 of linear equations vanishing on all vectors in a given subspace $V \subseteq E$, and the notion of the set U^0 of common solutions of a given set $U \subseteq E^*$ of linear equations.

The duality theorem (Theorem 8.1) shows that the dimensions of V and V^0 , and the dimensions of U and U^0 , are related in a crucial way.

It also shows that, in finite dimension, the maps $V \mapsto V^0$ and $U \mapsto U^0$ are inverse bijections from subspaces of E to subspaces of E^* .

Definition 8.2. Given a vector space E and its dual E^* , we say that a vector $v \in E$ and a linear form $u^* \in E^*$ are *orthogonal* iff $\langle u^*, v \rangle = 0$. Given a subspace V of E and a subspace U of E^* , we say that *V and U are orthogonal* iff $\langle u^*, v \rangle = 0$ for every $u^* \in U$ and every $v \in V$. Given a subset V of E (resp. a subset U of E^*), the *orthogonal V^0 of V* is the subspace V^0 of E^* defined such that

$$V^0 = \{u^* \in E^* \mid \langle u^*, v \rangle = 0, \text{ for every } v \in V\}$$

(resp. the *orthogonal U^0 of U* is the subspace U^0 of E defined such that

$$U^0 = \{v \in E \mid \langle u^*, v \rangle = 0, \text{ for every } u^* \in U\}).$$

The subspace $V^0 \subseteq E^*$ is also called the *annihilator* of V .

The subspace $U^0 \subseteq E$ annihilated by $U \subseteq E^*$ does not have a special name. It seems reasonable to call it the *linear subspace (or linear variety) defined by U* .

Informally, V^0 is the *set of linear equations that vanish on V* , and U^0 is *the set of common zeros of all linear equations in U* . We can also define V^0 by

$$V^0 = \{u^* \in E^* \mid V \subseteq \text{Ker } u^*\}$$

and U^0 by

$$U^0 = \bigcap_{u^* \in U} \text{Ker } u^*.$$

Observe that $E^0 = \{0\} = (0)$, and $\{0\}^0 = E^*$.

Furthermore, if $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, and if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$.

It can also be shown that that $V \subseteq V^{00}$ for every subspace V of E , and that $U \subseteq U^{00}$ for every subspace U of E^* .

We will see shortly that in finite dimension, we have

$$V = V^{00} \quad \text{and} \quad U = U^{00}.$$

Here are some examples. Let $E = M_2(\mathbb{R})$, the space of real 2×2 matrices, and let V be the subspace of $M_2(\mathbb{R})$ spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We check immediately that the subspace V consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

that is, all symmetric matrices.

The matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

in V satisfy the equation

$$a_{12} - a_{21} = 0,$$

and all scalar multiples of these equations, so V^0 is the subspace of E^* spanned by the linear form given by

$$u^*(a_{11}, a_{12}, a_{21}, a_{22}) = a_{12} - a_{21}.$$

By the duality theorem (Theorem 8.1) we have

$$\dim(V^0) = \dim(E) - \dim(V) = 4 - 3 = 1.$$

The above example generalizes to $E = M_n(\mathbb{R})$ for any $n \geq 1$, but this time, consider the space U of linear forms asserting that a matrix A is symmetric; these are the linear forms spanned by the $n(n - 1)/2$ equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i < j \leq n;$$

Note there are no constraints on diagonal entries, and half of the equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i \neq j \leq n$$

are redundant. It is easy to check that the equations (linear forms) for which $i < j$ are linearly independent.

To be more precise, let U be the space of linear forms in E^* spanned by the linear forms

$$\begin{aligned} u_{ij}^*(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}) \\ = a_{ij} - a_{ji}, \quad 1 \leq i < j \leq n. \end{aligned}$$

The dimension of U is $n(n-1)/2$. Then, the set U^0 of common solutions of these equations is the space $\mathbf{S}(n)$ of symmetric matrices.

By the duality theorem (Theorem 8.1), this space has dimension

$$\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}.$$

If $E = M_n(\mathbb{R})$, consider the subspace U of linear forms in E^* spanned by the linear forms

$$\begin{aligned} u_{ij}^*(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}) \\ &= a_{ij} + a_{ji}, \quad 1 \leq i < j \leq n \\ u_{ii}^*(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}) \\ &= a_{ii}, \quad 1 \leq i \leq n. \end{aligned}$$

It is easy to see that these linear forms are linearly independent, so $\dim(U) = n(n+1)/2$.

The space U^0 of matrices $A \in M_n(\mathbb{R})$ satisfying all of the above equations is clearly the space **Skew**(n) of skew-symmetric matrices.

By the duality theorem (Theorem 8.1), the dimension of U^0 is

$$\frac{n(n-1)}{2} = n^2 - \frac{n(n+1)}{2}.$$

For yet another example with $E = M_n(\mathbb{R})$, for any $A \in M_n(\mathbb{R})$, consider the linear form in E^* given by

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

called the *trace* of A .

The subspace U^0 of E consisting of all matrices A such that $\text{tr}(A) = 0$ is a space of dimension $n^2 - 1$.

The dimension equations

$$\begin{aligned}\dim(V) + \dim(V^0) &= \dim(E) \\ \dim(U) + \dim(U^0) &= \dim(E)\end{aligned}$$

are always true (if E is finite-dimensional). This is part of the duality theorem (Theorem 8.1).

In contrast with the previous examples, given a matrix $A \in M_n(\mathbb{R})$, the equations asserting that $A^\top A = I$ are not linear constraints.

For example, for $n = 2$, we have

$$\begin{aligned} a_{11}^2 + a_{21}^2 &= 1 \\ a_{21}^2 + a_{22}^2 &= 1 \\ a_{11}a_{12} + a_{21}a_{22} &= 0. \end{aligned}$$

Given a vector space E and any basis $(u_i)_{i \in I}$ for E , we can associate to each u_i a linear form $u_i^* \in E^*$, and the u_i^* have some remarkable properties.

Definition 8.3. Given a vector space E and any basis $(u_i)_{i \in I}$ for E , by Proposition 1.11, for every $i \in I$, there is a unique linear form u_i^* such that

$$u_i^*(u_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for every $j \in I$. The linear form u_i^* is called the *coordinate form* of index i w.r.t. the basis $(u_i)_{i \in I}$.

The reason for the terminology *coordinate form* was explained in Section 1.7.

We proved in Theorem 1.14 that if (u_1, \dots, u_n) is a basis of E , then (u_1^*, \dots, u_n^*) is a basis of E^* called the *dual basis*.

If (u_1, \dots, u_n) is a basis of \mathbb{R}^n (more generally K^n), it is possible to find explicitly the dual basis (u_1^*, \dots, u_n^*) , where each u_i^* is represented by a row vector.

For example, consider the columns of the Bézier matrix

$$B_4 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The form u_1^* is represented by a row vector $(\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4)$ such that

$$(\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4) \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1 \ 0 \ 0 \ 0).$$

This implies that u_1^* is the first row of the inverse of B_4 .

Since

$$B_4^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the linear forms $(u_1^*, u_2^*, u_3^*, u_4^*)$ correspond to the rows of B_4^{-1} .

In particular, u_1^* is represented by $(1 \ 1 \ 1 \ 1)$.

The above method works for any n . Given any basis (u_1, \dots, u_n) of \mathbb{R}^n , if P is the $n \times n$ matrix whose j th column is u_j , then the dual form u_i^* is given by the i th row of the matrix P^{-1} .

We have the following important duality theorem adapted from E. Artin.

8.3 The Duality Theorem

Theorem 8.1. (*Duality theorem*) *Let E be a vector space of dimension n . The following properties hold:*

- (a) *For every basis (u_1, \dots, u_n) of E , the family of coordinate forms (u_1^*, \dots, u_n^*) is a basis of E^* .*
- (b) *For every subspace V of E , we have $V^{00} = V$.*
- (c) *For every pair of subspaces V and W of E such that $E = V \oplus W$, with V of dimension m , for every basis (u_1, \dots, u_n) of E such that (u_1, \dots, u_m) is a basis of V and (u_{m+1}, \dots, u_n) is a basis of W , the family (u_1^*, \dots, u_m^*) is a basis of the orthogonal W^0 of W in E^* . Furthermore, we have $W^{00} = W$, and*

$$\dim(W) + \dim(W^0) = \dim(E).$$

- (d) *For every subspace U of E^* , we have*

$$\dim(U) + \dim(U^0) = \dim(E),$$

where U^0 is the orthogonal of U in E , and $U^{00} = U$.

Part (a) of Theorem 8.1 shows that

$$\dim(E) = \dim(E^*),$$

and if (u_1, \dots, u_n) is a basis of E , then (u_1^*, \dots, u_n^*) is a basis of the dual space E^* called the *dual basis* of (u_1, \dots, u_n) .

Define the function \mathcal{E} (\mathcal{E} for *equations*) from subspaces of E to subspaces of E^* and the function \mathcal{Z} (\mathcal{Z} for *zeros*) from subspaces of E^* to subspaces of E by

$$\begin{aligned}\mathcal{E}(V) &= V^0, & V &\subseteq E \\ \mathcal{Z}(U) &= U^0, & U &\subseteq E^*.\end{aligned}$$

By part (c) and (d) of theorem 8.1,

$$\begin{aligned}(\mathcal{Z} \circ \mathcal{E})(V) &= V^{00} = V \\ (\mathcal{E} \circ \mathcal{Z})(U) &= U^{00} = U,\end{aligned}$$

so $\mathcal{Z} \circ \mathcal{E} = \text{id}$ and $\mathcal{E} \circ \mathcal{Z} = \text{id}$, and the maps \mathcal{E} and \mathcal{V} are inverse bijections.

These maps set up a *duality* between subspaces of E , and subspaces of E^* .



One should be careful that this bijection does not hold if E has infinite dimension. Some restrictions on the dimensions of U and V are needed.

Suppose that V is a subspace of \mathbb{R}^n of dimension m and that (v_1, \dots, v_m) is a basis of V .

To find a basis of V^0 , we first extend (v_1, \dots, v_m) to a basis (v_1, \dots, v_n) of \mathbb{R}^n , and then by part (c) of Theorem 8.1, we know that $(v_{m+1}^*, \dots, v_n^*)$ is a basis of V^0 .

For example, suppose that V is the subspace of \mathbb{R}^4 spanned by the two linearly independent vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

the first two vectors of the Haar basis in \mathbb{R}^4 .

The four columns of the Haar matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

form a basis of \mathbb{R}^4 , and the inverse of W is given by

$$W^{-1} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix}.$$

Since the dual basis $(v_1^*, v_2^*, v_3^*, v_4^*)$ is given by the row of W^{-1} , the last two rows of W^{-1} ,

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix},$$

form a basis of V^0 .

We also obtain a basis by rescaling by the factor $1/2$, so the linear forms given by the row vectors

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

form a basis of V^0 , the space of linear forms (linear equations) that vanish on the subspace V .

The method that we described to find V^0 requires first extending a basis of V and then inverting a matrix, but there is a more direct method.

Indeed, let A be the $n \times m$ matrix whose columns are the basis vectors (v_1, \dots, v_m) of V . Then, a linear form u represented by a row vector belongs to V^0 iff $uv_i = 0$ for $i = 1, \dots, m$ iff

$$uA = 0$$

iff

$$A^\top u^\top = 0.$$

Therefore, all we need to do is to *find a basis of the nullspace of A^\top* .

This can be done quite effectively using the reduction of a matrix to reduced row echelon form (rref); see Section 5.9.

Here is another example illustrating the power of Theorem 8.1.

Let $E = M_n(\mathbb{R})$, and consider the equations asserting that the sum of the entries in every row of a matrix $A \in M_n(\mathbb{R})$ is equal to the same number.

We have $n - 1$ equations

$$\sum_{j=1}^n (a_{ij} - a_{i+1j}) = 0, \quad 1 \leq i \leq n - 1,$$

and it is easy to see that they are linearly independent.

Therefore, the space U of linear forms in E^* spanned by the above linear forms (equations) has dimension $n - 1$, and the space U^0 of matrices satisfying all these equations has dimension $n^2 - n + 1$.

It is not so obvious to find a basis for this space.

When E is of finite dimension n and (u_1, \dots, u_n) is a basis of E , we noted that the family (u_1^*, \dots, u_n^*) is a basis of the dual space E^* ,

Let us see how the coordinates of a linear form $\varphi^* \in E^*$ over the basis (u_1^*, \dots, u_n^*) vary under a change of basis.

Let (u_1, \dots, u_n) and (v_1, \dots, v_n) be two bases of E , and let $P = (a_{ij})$ be the change of basis matrix from (u_1, \dots, u_n) to (v_1, \dots, v_n) , so that

$$v_j = \sum_{i=1}^n a_{ij} u_i.$$

If

$$\varphi^* = \sum_{i=1}^n \varphi_i u_i^* = \sum_{i=1}^n \varphi'_i v_i^*,$$

after some algebra, we get

$$\varphi'_j = \sum_{i=1}^n a_{ij} \varphi_i.$$

Comparing with the change of basis

$$v_j = \sum_{i=1}^n a_{ij} u_i,$$

we note that this time, the coordinates (φ_i) of the linear form φ^* change in the *same direction* as the change of basis.

For this reason, we say that the coordinates of linear forms are *covariant*.

By abuse of language, it is often said that linear forms are *covariant*, which explains why the term *covector* is also used for a linear form.

Observe that if (e_1, \dots, e_n) is a basis of the vector space E , then, as a linear map from E to K , every linear form $f \in E^*$ is represented by a $1 \times n$ matrix, that is, by a *row vector*

$$(\lambda_1 \cdots \lambda_n),$$

with respect to the basis (e_1, \dots, e_n) of E , and 1 of K , where $f(e_i) = \lambda_i$.

A vector $u = \sum_{i=1}^n u_i e_i \in E$ is represented by a $n \times 1$ matrix, that is, by a *column vector*

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

and the action of f on u , namely $f(u)$, is represented by the matrix product

$$(\lambda_1 \ \cdots \ \lambda_n) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \lambda_1 u_1 + \cdots + \lambda_n u_n.$$

On the other hand, with respect to the dual basis (e_1^*, \dots, e_n^*) of E^* , the linear form f is represented by the column vector

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

We will now pin down the relationship between a vector space E and its bidual E^{**} .

Proposition 8.2. *Let E be a vector space. The following properties hold:*

(a) *The linear map $\text{eval}_E: E \rightarrow E^{**}$ defined such that*

$$\text{eval}_E(v) = \text{eval}_v, \quad \text{for all } v \in E,$$

that is, $\text{eval}_E(v)(u^) = \langle u^*, v \rangle = u^*(v)$ for every $u^* \in E^*$, is injective.*

(b) *When E is of finite dimension n , the linear map $\text{eval}_E: E \rightarrow E^{**}$ is an isomorphism (called the canonical isomorphism).*

When E is of finite dimension and (u_1, \dots, u_n) is a basis of E , in view of the canonical isomorphism $\text{eval}_E: E \rightarrow E^{**}$, the basis $(u_1^{**}, \dots, u_n^{**})$ of the bidual is identified with (u_1, \dots, u_n) .

Proposition 8.2 can be reformulated very fruitfully in terms of pairings.

Definition 8.4. Given two vector spaces E and F over K , a *pairing between E and F* is a bilinear map $\varphi: E \times F \rightarrow K$. Such a pairing is *nondegenerate* iff

- (1) for every $u \in E$, if $\varphi(u, v) = 0$ for all $v \in F$, then $u = 0$, and
- (2) for every $v \in F$, if $\varphi(u, v) = 0$ for all $u \in E$, then $v = 0$.

A pairing $\varphi: E \times F \rightarrow K$ is often denoted by $\langle -, - \rangle: E \times F \rightarrow K$.

For example, the map $\langle -, - \rangle: E^* \times E \rightarrow K$ defined earlier is a nondegenerate pairing (use the proof of (a) in Proposition 8.2).

Given a pairing $\varphi: E \times F \rightarrow K$, we can define two maps $l_\varphi: E \rightarrow F^*$ and $r_\varphi: F \rightarrow E^*$ as follows:

For every $u \in E$, we define the linear form $l_\varphi(u)$ in F^* such that

$$l_\varphi(u)(y) = \varphi(u, y) \quad \text{for every } y \in F,$$

and for every $v \in F$, we define the linear form $r_\varphi(v)$ in E^* such that

$$r_\varphi(v)(x) = \varphi(x, v) \quad \text{for every } x \in E.$$

Proposition 8.3. *Given two vector spaces E and F over K , for every nondegenerate pairing $\varphi: E \times F \rightarrow K$ between E and F , the maps $l_\varphi: E \rightarrow F^*$ and $r_\varphi: F \rightarrow E^*$ are linear and injective. Furthermore, if E and F have finite dimension, then this dimension is the same and $l_\varphi: E \rightarrow F^*$ and $r_\varphi: F \rightarrow E^*$ are bijections.*

When E has finite dimension, the nondegenerate pairing $\langle -, - \rangle: E^* \times E \rightarrow K$ yields another proof of the existence of a natural isomorphism between E and E^{**} .

Interesting nondegenerate pairings arise in exterior algebra.

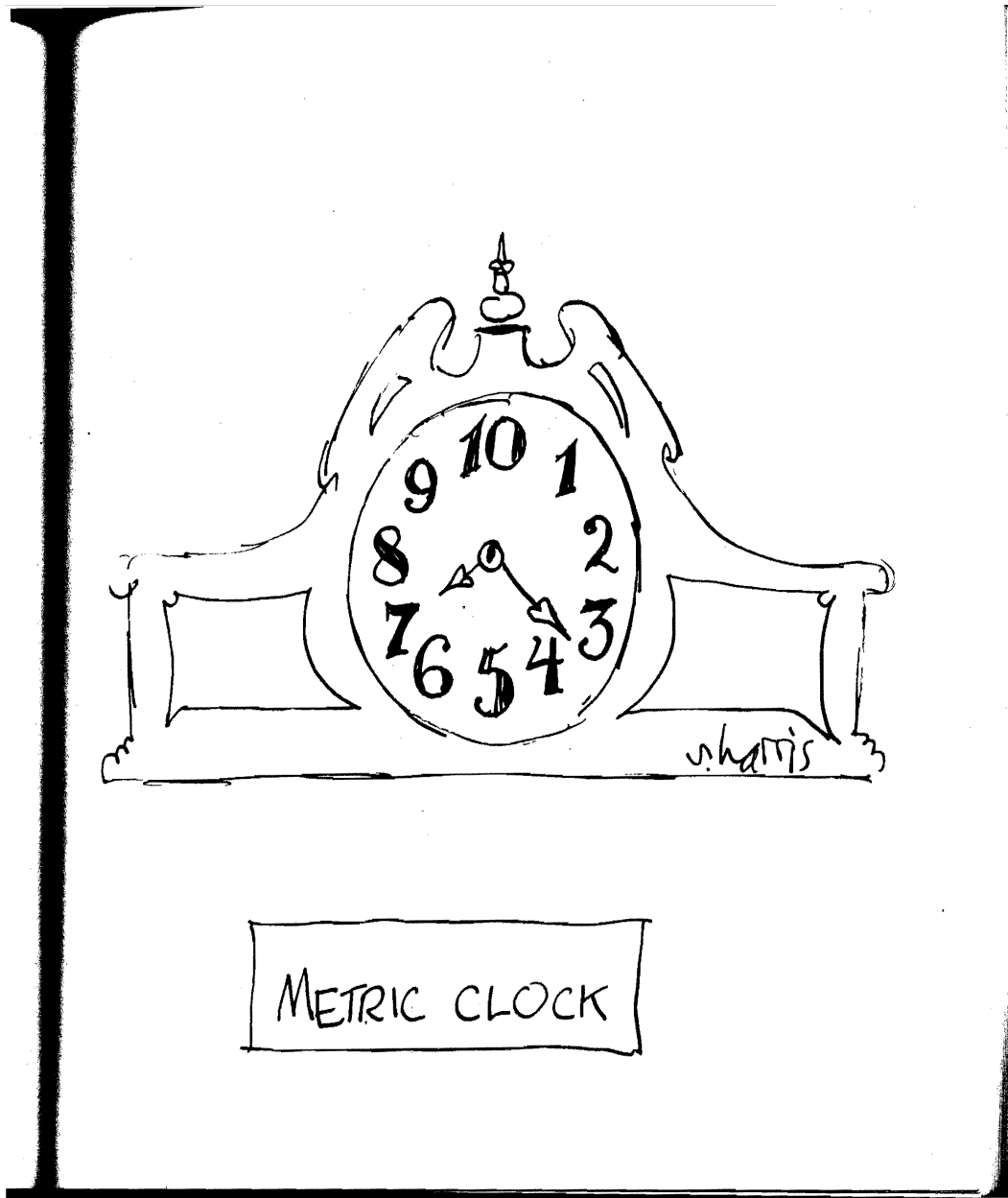


Figure 8.1: Metric Clock

8.4 Hyperplanes and Linear Forms

Actually, Proposition 8.4 below follows from parts (c) and (d) of Theorem 8.1, but we feel that it is also interesting to give a more direct proof.

Proposition 8.4. *Let E be a vector space. The following properties hold:*

- (a) *Given any nonnull linear form $f^* \in E^*$, its kernel $H = \text{Ker } f^*$ is a hyperplane.*
- (b) *For any hyperplane H in E , there is a (nonnull) linear form $f^* \in E^*$ such that $H = \text{Ker } f^*$.*
- (c) *Given any hyperplane H in E and any (nonnull) linear form $f^* \in E^*$ such that $H = \text{Ker } f^*$, for every linear form $g^* \in E^*$, $H = \text{Ker } g^*$ iff $g^* = \lambda f^*$ for some $\lambda \neq 0$ in K .*

We leave as an exercise the fact that every subspace $V \neq E$ of a vector space E , is the intersection of all hyperplanes that contain V .

We now consider the notion of transpose of a linear map and of a matrix.

8.5 Transpose of a Linear Map and of a Matrix

Given a linear map $f: E \rightarrow F$, it is possible to define a map $f^\top: F^* \rightarrow E^*$ which has some interesting properties.

Definition 8.5. Given a linear map $f: E \rightarrow F$, the *transpose* $f^\top: F^* \rightarrow E^*$ of f is the linear map defined such that

$$f^\top(v^*) = v^* \circ f,$$

for every $v^* \in F^*$, as shown in the diagram below:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ & \searrow f^\top(v^*) & \downarrow v^* \\ & & K. \end{array}$$

Equivalently, the linear map $f^\top: F^* \rightarrow E^*$ is defined such that

$$\langle v^*, f(u) \rangle = \langle f^\top(v^*), u \rangle,$$

for all $u \in E$ and all $v^* \in F^*$.

It is easy to verify that the following properties hold:

$$\begin{aligned}(f + g)^\top &= f^\top + g^\top \\ (g \circ f)^\top &= f^\top \circ g^\top \\ \text{id}_E^\top &= \text{id}_{E^*}.\end{aligned}$$



Note the reversal of composition on the right-hand side of $(g \circ f)^\top = f^\top \circ g^\top$.

The equation $(g \circ f)^\top = f^\top \circ g^\top$ implies the following useful proposition.

Proposition 8.5. *If $f: E \rightarrow F$ is any linear map, then the following properties hold:*

- (1) *If f is injective, then f^\top is surjective.*
- (2) *If f is surjective, then f^\top is injective.*

We also have the following property showing the naturality of the eval map.

Proposition 8.6. *For any linear map $f: E \rightarrow F$, we have*

$$f^{\top\top} \circ \text{eval}_E = \text{eval}_F \circ f,$$

or equivalently, the following diagram commutes:

$$\begin{array}{ccc} E^{**} & \xrightarrow{f^{\top\top}} & F^{**} \\ \text{eval}_E \uparrow & & \uparrow \text{eval}_F \\ E & \xrightarrow{f} & F. \end{array}$$

If E and F are finite-dimensional, then eval_E and eval_F are isomorphisms, so Proposition 8.6 shows that

$$f^{\top\top} = \text{eval}_F^{-1} \circ f \circ \text{eval}_E. \quad (*)$$

The above equation is often interpreted as follows: if we identify E with its bidual E^{**} and F with its bidual F^{**} , then $f^{\top\top} = f$.

This is an abuse of notation; the rigorous statement is (*).

Proposition 8.7. *Given a linear map $f: E \rightarrow F$, for any subspace V of E , we have*

$$f(V)^0 = (f^\top)^{-1}(V^0) = \{w^* \in F^* \mid f^\top(w^*) \in V^0\}.$$

As a consequence,

$$\text{Ker } f^\top = (\text{Im } f)^0 \quad \text{and} \quad \text{Ker } f = (\text{Im } f^\top)^0.$$

The following theorem shows the relationship between the rank of f and the rank of f^\top .

Theorem 8.8. *Given a linear map $f: E \rightarrow F$, the following properties hold.*

(a) *The dual $(\text{Im } f)^*$ of $\text{Im } f$ is isomorphic to $\text{Im } f^\top = f^\top(F^*)$; that is,*

$$(\text{Im } f)^* \approx \text{Im } f^\top.$$

(b) *If F is finite dimensional, then $\text{rk}(f) = \text{rk}(f^\top)$.*

The following proposition can be shown, but it requires a generalization of the duality theorem.

Proposition 8.9. *If $f: E \rightarrow F$ is any linear map, then the following identities hold:*

$$\begin{aligned} \text{Im } f^\top &= (\text{Ker } (f))^0 \\ \text{Ker } (f^\top) &= (\text{Im } f)^0 \\ \text{Im } f &= (\text{Ker } (f^\top))^0 \\ \text{Ker } (f) &= (\text{Im } f^\top)^0. \end{aligned}$$

The following proposition shows the relationship between the matrix representing a linear map $f: E \rightarrow F$ and the matrix representing its transpose $f^\top: F^* \rightarrow E^*$.

Proposition 8.10. *Let E and F be two vector spaces, and let (u_1, \dots, u_n) be a basis for E , and (v_1, \dots, v_m) be a basis for F . Given any linear map $f: E \rightarrow F$, if $M(f)$ is the $m \times n$ -matrix representing f w.r.t. the bases (u_1, \dots, u_n) and (v_1, \dots, v_m) , the $n \times m$ -matrix $M(f^\top)$ representing $f^\top: F^* \rightarrow E^*$ w.r.t. the dual bases (v_1^*, \dots, v_m^*) and (u_1^*, \dots, u_n^*) is the transpose $M(f)^\top$ of $M(f)$.*

We now can give a very short proof of the fact that the rank of a matrix is equal to the rank of its transpose.

Proposition 8.11. *Given a $m \times n$ matrix A over a field K , we have $\text{rk}(A) = \text{rk}(A^\top)$.*

Thus, given an $m \times n$ -matrix A , the maximum number of linearly independent columns is equal to the maximum number of linearly independent rows.

Proposition 8.12. *Given any $m \times n$ matrix A over a field K (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of A is the maximum natural number r such that there is an invertible $r \times r$ submatrix of A obtained by selecting r rows and r columns of A .*

For example, the 3×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

has rank 2 iff one of the three 2×2 matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \quad \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

is invertible. We saw in Chapter 4 that this is equivalent to the fact the determinant of one of the above matrices is nonzero.

This is not a very efficient way of finding the rank of a matrix. We will see that there are better ways using various decompositions such as LU, QR, or SVD.

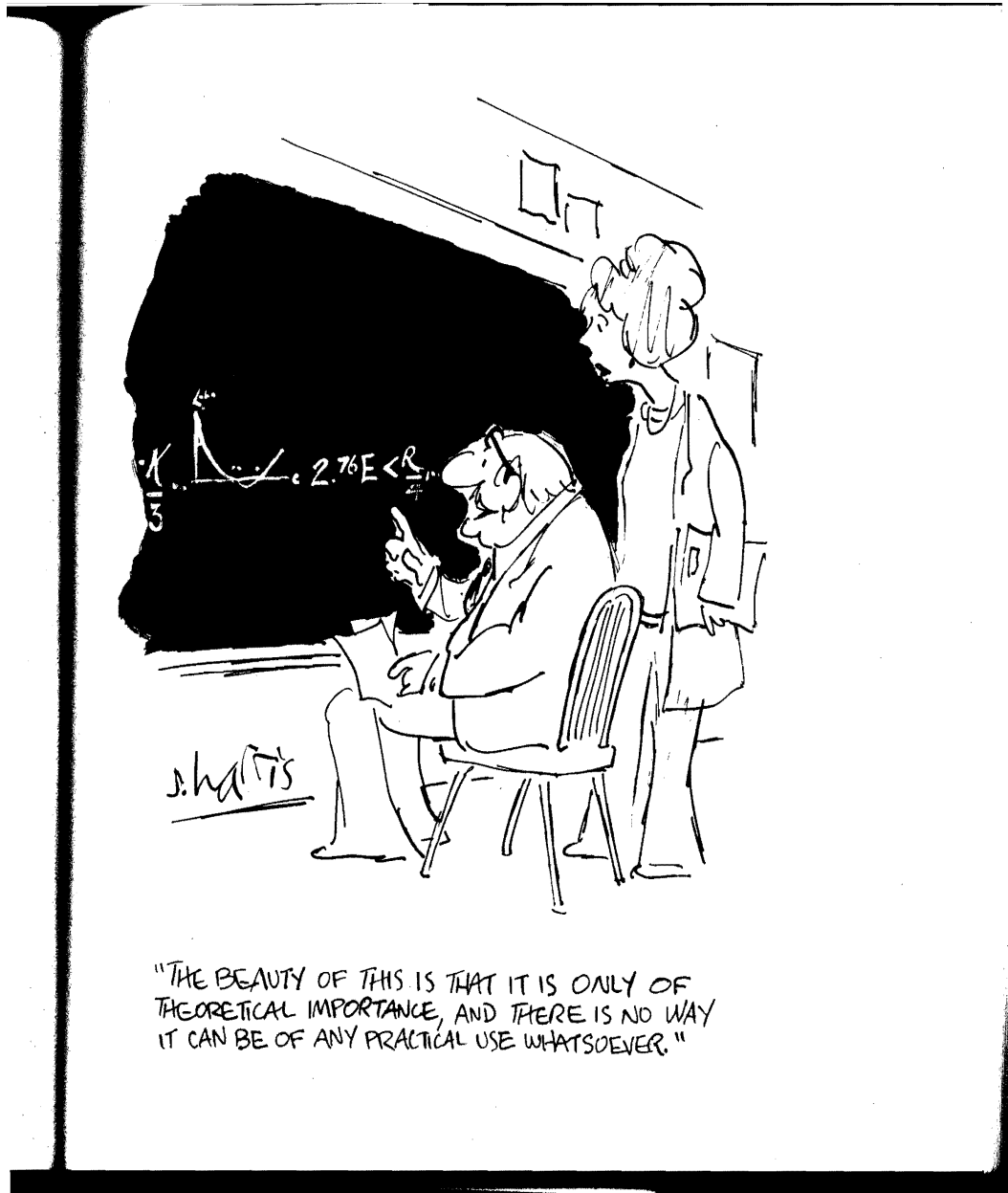


Figure 8.2: Beauty

8.6 The Four Fundamental Subspaces

Given a linear map $f: E \rightarrow F$ (where E and F are finite-dimensional), Proposition 8.7 revealed that the four spaces

$$\text{Im } f, \text{Im } f^\top, \text{Ker } f, \text{Ker } f^\top$$

play a special role. They are often called the *fundamental subspaces* associated with f .

These spaces are related in an intimate manner, since Proposition 8.7 shows that

$$\begin{aligned}\text{Ker } f &= (\text{Im } f^\top)^0 \\ \text{Ker } f^\top &= (\text{Im } f)^0,\end{aligned}$$

and Theorem 8.8 shows that

$$\text{rk}(f) = \text{rk}(f^\top).$$

It is instructive to translate these relations in terms of matrices (actually, certain linear algebra books make a big deal about this!).

If $\dim(E) = n$ and $\dim(F) = m$, given any basis (u_1, \dots, u_n) of E and a basis (v_1, \dots, v_m) of F , we know that f is represented by an $m \times n$ matrix $A = (a_{ij})$, where the j th column of A is equal to $f(u_j)$ over the basis (v_1, \dots, v_m) .

Furthermore, the transpose map f^\top is represented by the $n \times m$ matrix A^\top (with respect to the dual bases).

Consequently, the four fundamental spaces

$$\text{Im } f, \text{Im } f^\top, \text{Ker } f, \text{Ker } f^\top$$

correspond to

- (1) The *column space* of A , denoted by $\text{Im } A$ or $\mathcal{R}(A)$; this is the subspace of \mathbb{R}^m spanned by the columns of A , which corresponds to the image $\text{Im } f$ of f .
- (2) The *kernel* or *nullspace* of A , denoted by $\text{Ker } A$ or $\mathcal{N}(A)$; this is the subspace of \mathbb{R}^n consisting of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.
- (3) The *row space* of A , denoted by $\text{Im } A^\top$ or $\mathcal{R}(A^\top)$; this is the subspace of \mathbb{R}^n spanned by the rows of A , or equivalently, spanned by the columns of A^\top , which corresponds to the image $\text{Im } f^\top$ of f^\top .
- (4) The *left kernel* or *left nullspace* of A denoted by $\text{Ker } A^\top$ or $\mathcal{N}(A^\top)$; this is the kernel (nullspace) of A^\top , the subspace of \mathbb{R}^m consisting of all vectors $y \in \mathbb{R}^m$ such that $A^\top y = 0$, or equivalently, $y^\top A = 0$.

Recall that the dimension r of $\text{Im } f$, which is also equal to the dimension of the column space $\text{Im } A = \mathcal{R}(A)$, is the *rank* of A (and f).

Then, some of our previous results can be reformulated as follows:

1. The column space $\mathcal{R}(A)$ of A has dimension r .
2. The nullspace $\mathcal{N}(A)$ of A has dimension $n - r$.
3. The row space $\mathcal{R}(A^\top)$ has dimension r .
4. The left nullspace $\mathcal{N}(A^\top)$ of A has dimension $m - r$.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part I* (see Strang [32]).

The two statements

$$\begin{aligned}\text{Ker } f &= (\text{Im } f^\top)^0 \\ \text{Ker } f^\top &= (\text{Im } f)^0\end{aligned}$$

translate to

- (1) The nullspace of A is the orthogonal of the row space of A .
- (2) The left nullspace of A is the orthogonal of the column space of A .

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part II* (see Strang [32]).

Since vectors are represented by column vectors and linear forms by row vectors (over a basis in E or F), a vector $x \in \mathbb{R}^n$ is orthogonal to a linear form y if

$$yx = 0.$$

Then, a vector $x \in \mathbb{R}^n$ is orthogonal to the row space of A iff x is orthogonal to every row of A , namely $Ax = 0$, which is equivalent to the fact that x belong to the nullspace of A .

Similarly, the column vector $y \in \mathbb{R}^m$ (representing a linear form over the dual basis of F^*) belongs to the nullspace of A^\top iff $A^\top y = 0$, iff $y^\top A = 0$, which means that the linear form given by y^\top (over the basis in F) is orthogonal to the column space of A .

Since (2) is equivalent to the fact that *the column space of A is equal to the orthogonal of the left nullspace of A* , we get the following criterion for the solvability of an equation of the form $Ax = b$:

The equation $Ax = b$ has a solution iff for all $y \in \mathbb{R}^m$, if $A^\top y = 0$, then $y^\top b = 0$.

Indeed, the condition on the right-hand side says that b is orthogonal to the left nullspace of A , that is, that b belongs to the column space of A .

This criterion can be cheaper to check than checking directly that b is spanned by the columns of A . For example, if we consider the system

$$\begin{aligned}x_1 - x_2 &= b_1 \\x_2 - x_3 &= b_2 \\x_3 - x_1 &= b_3\end{aligned}$$

which, in matrix form, is written $Ax = b$ as below:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we see that the rows of the matrix A add up to 0.

In fact, it is easy to convince ourselves that the left nullspace of A is spanned by $y = (1, 1, 1)$, and so the system is solvable iff $y^\top b = 0$, namely

$$b_1 + b_2 + b_3 = 0.$$

Note that the above criterion can also be stated negatively as follows:

The equation $Ax = b$ has no solution iff there is some $y \in \mathbb{R}^m$ such that $A^\top y = 0$ and $y^\top b \neq 0$.

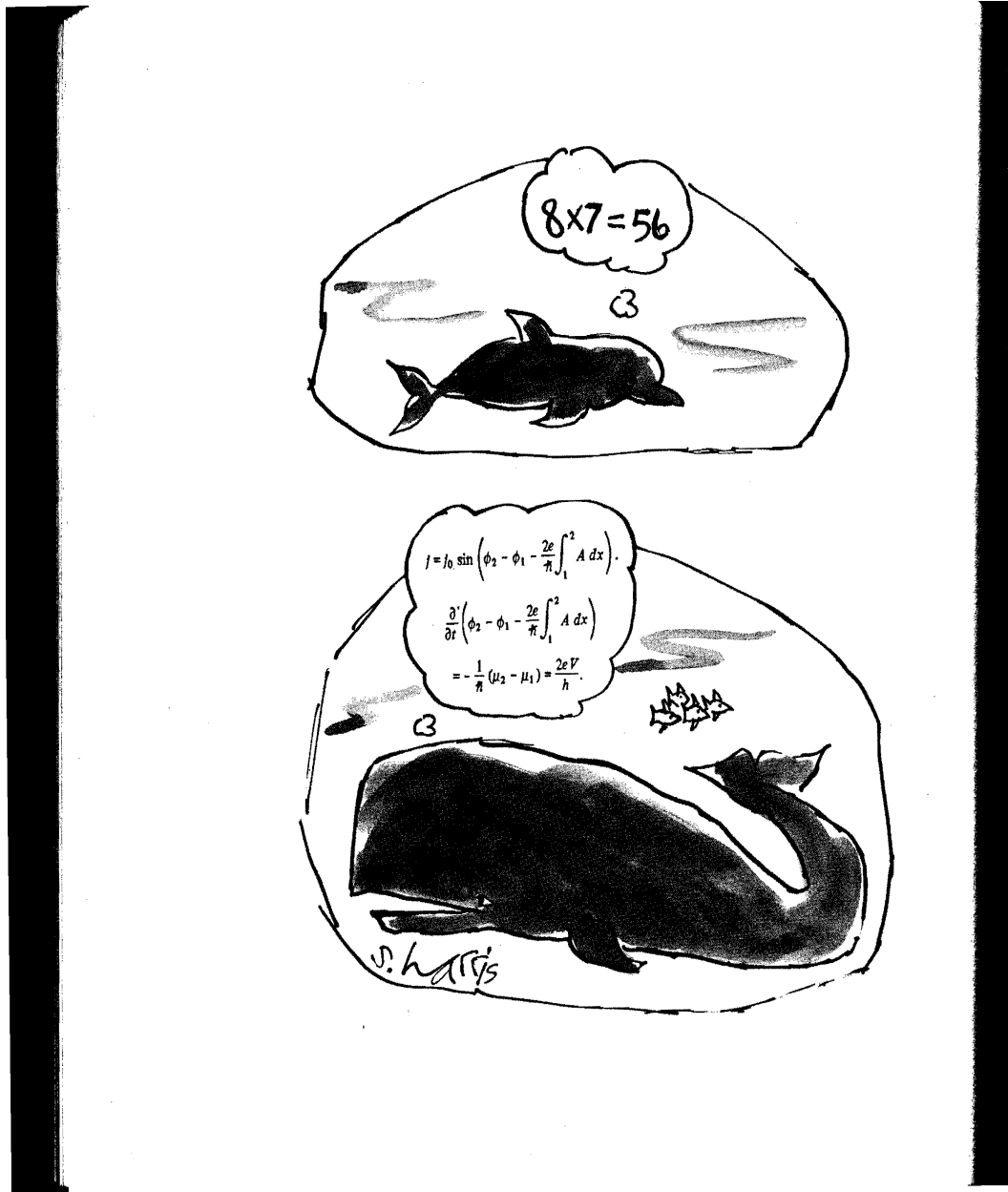


Figure 8.3: Brain Size?

