CIS1921

Lecture 8: Intro to Constraint Programming

Logistics

- No OH over break; will resume on 3/18
  - Voluntary OH might be held (check Ed)
- Final project partners due **tomorrow at 11:59pm**
- Final project proposals due **3/20**
- **Late Days NOT ALLOWED on final project**
- Homework 3 due on **3/18**
- **Guest Lecture on 3/20…**

LAURENT PERRON

- Tech lead of the Operations Research team at Google.
- One of the main contributors to OR-Tools!

Warm Up

What is Uber, and how does it work?

Constraints

- Recall: many decision problems involve checking if there is a solution that satisfies certain constraints
- A **constraint** is just a rule that limits which possible solutions are acceptable
- **Ex:** CNF-SAT
  - Solution: a truth assignment
  - Constraints: in each clause, at least one variable is assigned to True

Constraint Satisfaction

- A **constraint satisfaction problem** is defined by:
  - a set of **variables**, each with its own range of **values**
  - a set of **constraints**
- A **candidate solution** is any assignment of vars to values
- Candidate solutions that satisfy all constraints are **feasible**

Constraint Programming

- "Like LP, but with more complex constraints"
- OR-Tools has a new constraint programming solver called **CP-SAT**
- Behind the scenes: turns constraints into clauses, then uses SAT solver!
  - vast oversimplification…

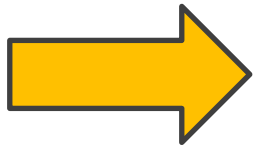
Results of Minizinc CP Challenge 2021

Category	Gold	Silver
Fixed	SICStus Prolog	JaCoP
Free	OR-Tools	PicatSAT
Parallel	OR-Tools	PicatSAT
Open	OR-Tools	sunny-cp
Local Search	Yuck	OscaR/CBLS

Cryptarithms

- In a cryptarithmetic puzzle, want to replace each letter with a different digit to make the arithmetic valid
  - no leading zeros

```
    A B
+   A
-------
  B C C
```



```
    9 1
+   9
-------
  1 0 0
```

Your Turn

Classic Example: Cryptarithms

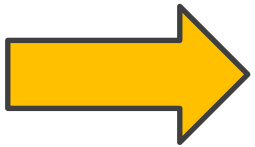
- In a cryptarithmetic puzzle, want to replace each letter with a different digit to make the arithmetic valid
  - no leading zeros

```
    S E N D
+   M O R E
-----------
  M O N E Y
```

Classic Example: Cryptarithms

- In a cryptarithmetic puzzle, want to replace each letter with a <u>different</u> digit to make the arithmetic valid
  - no leading zeros

```
    S E N D          9 5 6 7
+   M O R E    +     1 0 8 5
-----------    -------------
  M O N E Y        1 0 6 5 2
```



Classic Example: Cryptarithms

Constraint program:

- Variables for each letter, most with range [0...9]
  - S, M have range [1...9], since no leading zeros
- Constraint 1: the arithmetic expression holds
- Constraint 2: all vars have different value

```
      S E N D
+     M O R E
-------------
  M O N E Y
```

CP-SAT Documentation

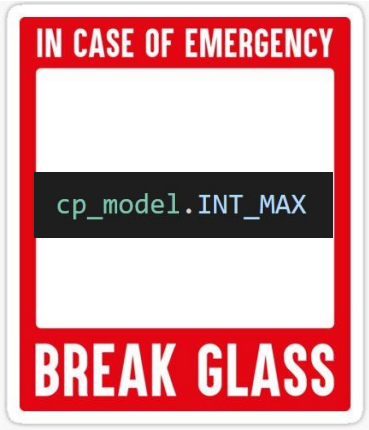
- For reference (variables, constraints):

https://developers.google.com/optimization/cp/cp_solver

Basic Variables in CP-SAT

- `model.NewIntVar(lower_bnd, upper_bnd, name)`
- `model.NewBoolVar(name)`
  - Equivalent to `model.NewIntVar(0, 1, name)`

- Returns newly created variable (just like MIP)
- CP-SAT **only** works over discrete, finite domains
  - No `NumVars`, integers only!
  - No infinite bounds



Linear Constraints in CP-SAT

- Adding/scalar multiplying vars gives a (linear) **expression**
- Linear expr. with an (in)equality gives a **linear constraint**
  - Unlike MIP, we can also use not equals (`!=`)
- Unlike MIP, coefficients **must** also be integers
  - If you have fractional coefficients, you need to scale them up to integers or use MIP solver instead

- `model.Add(linear_constraint)`

Solving MIP with CP-SAT

Suppose we had the following MIP:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 5 \\ & 2x_1 - 3x_2 \leq 4 \end{aligned}$$

Solving MIP with CP-SAT

Suppose we had the following MIP:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 5 \\ & 2x_1 - 3x_2 \leq 4 \end{aligned}$$

Using a MIP solver, the optimal solution is:

$$x_1 = 3.8 \quad x_2 = 1.2$$

CP-SAT won't return this, because we have to define x_1 and x_2 as intvars

Solving MIP with CP-SAT

modify our MIP by scaling *the constants of the inequality* by a factor of 10 (or a higher power of 10):

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 50 \\ & 2x_1 - 3x_2 \leq 40 \end{aligned}$$

Basic Nonlinear Constraints

- `model.AddMultiplicationEquality(target, [v1,v2, …, vn])`
  - Adds constraint: `target == v1 * v2 * ... * vn`
- `model.AddMaxEquality(target, var_arr)`
  - Adds constraint: `target == Max(var_arr)`

The AllDifferent Constraints

- `model.AddAllDifferent(var_arr)`

- Forces all vars in the array to take on different values!
- Very common in practice
  - Esp. for assignment problems, scheduling, etc.

Classic Example: Cryptarithms

Constraint program:

- Variables for each letter, most with range [0...9]
  - S, M have range [1...9], since no leading zeros
- Constraint 1: the arithmetic expression holds
- Constraint 2: all vars have different value

```
      S E N D
+     M O R E
-------------
  M O N E Y
```

Cryptarithms in OR-Tools

- Initializing the model and declaring variables

```
from ortools.sat.python import cp_model

model = cp_model.CpModel()
S = model.NewIntVar(1, 9, 'S')
E = model.NewIntVar(0, 9, 'E')
N = model.NewIntVar(0, 9, 'N')
D = model.NewIntVar(0, 9, 'D')
M = model.NewIntVar(1, 9, 'M')
O = model.NewIntVar(0, 9, 'O')
R = model.NewIntVar(0, 9, 'R')
Y = model.NewIntVar(0, 9, 'Y')
```

```
    S E N D
+   M O R E
-----------
  M O N E Y
```

Cryptarithms in OR-Tools

- Add arithmetic and all different constraints (yes, that easy!)

```
model.Add(
                    1000*S + 100*E + 10*N + D
  +                 1000*M + 100*O + 10*R + E
  ==   10000*M + 1000*O + 100*N + 10*E + Y
)
model.AddAllDifferent([S,E,N,D,M,O,R,Y])
```

```
      S E N D
+     M O R E
-------------
  M O N E Y
```

Cryptarithms in OR-Tools

- Solve and print the solution

```
solver = cp_model.CpSolver()
if solver.Solve(model) == cp_model.OPTIMAL:
    print([f'{v}={solver.Value(v)}' for v in [S,E,N,D,M,O,R,Y]])
```

- Output:

```
['S=9', 'E=5', 'N=6', 'D=7', 'M=1', 'O=0', 'R=8', 'Y=2']
```

```
    S E N D
+   M O R E
-----------
  M O N E Y
```

Optimization with CP-SAT

- We can also maximize/minimize an expression, e.g.

```
model.Maximize(7*a + b)
```

```
model.Minimize(
    sum(x[i] for i in range(10))
)
```

The Element Constraint

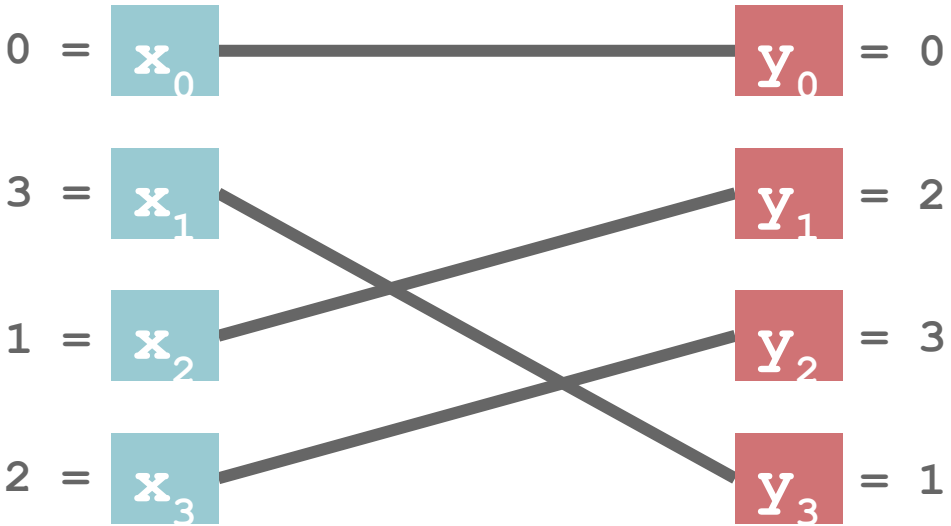
- `model.AddElement(index, var_arr, target)`
- Adds constraint: `target == var_arr[index]`
- Useful because `index` can be a variable
- The `var_arr` can also contain constants!

The Inverse Constraint

- `model.AddInverse(var_arr, inv_arr)`
- The arrays should have the same size n (can't use dicts)
- The vars in both arrays can only take values from 0 to $n-1$
- Adds the following constraints:
  - If `var_arr[i] == j`, then `inv_arr[j] == i`
  - If `inv_arr[j] == i`, then `var_arr[i] == j`

The Inverse Constraint

- `model.AddInverse([x0,x1,x2,x3], [y0,y1,y2,y3])`



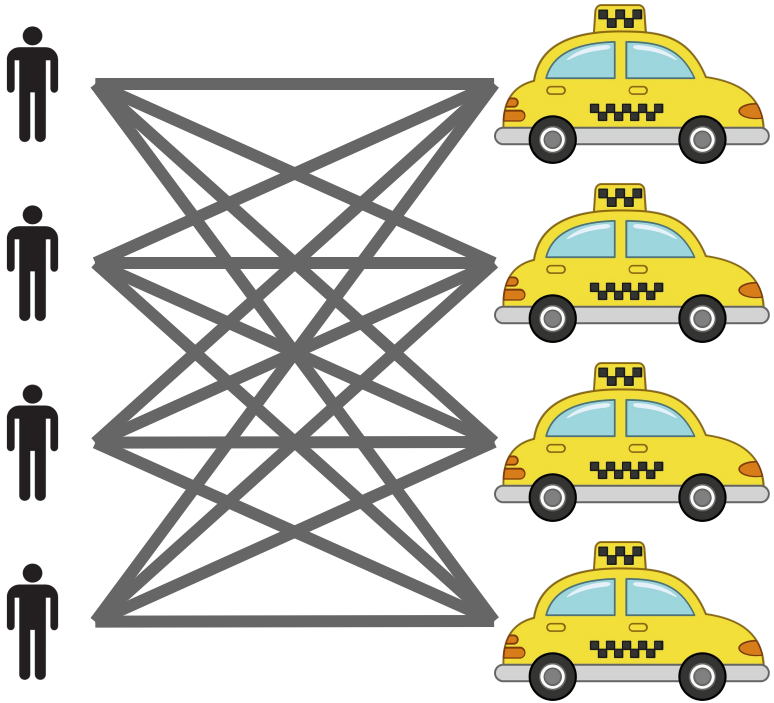
Equivalent to creating a "perfect matching" between two sets of variables.

Ex: Taxi Assignment

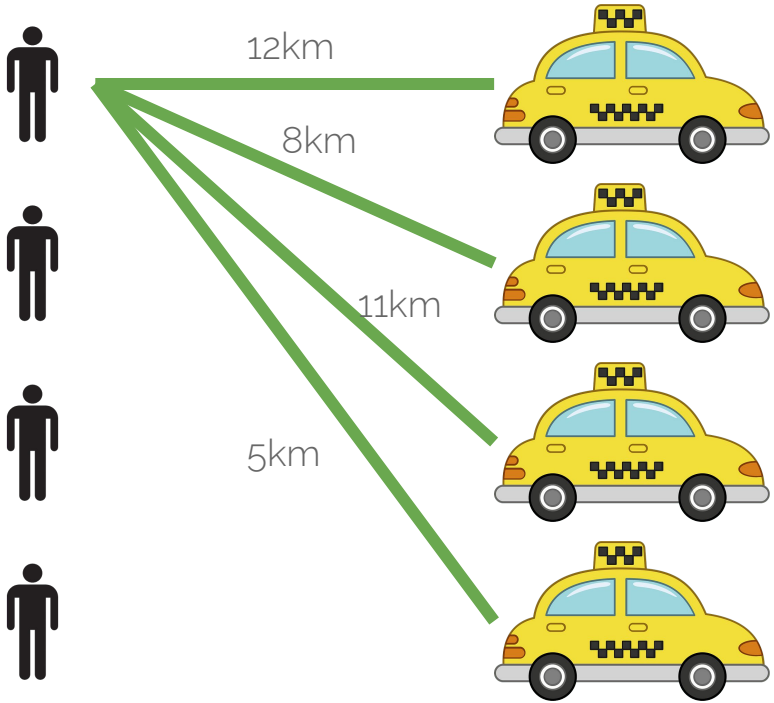
- A taxi service has n customers waiting for pickup
- There are n taxis available, one for each customer
- We know the distance between each taxi and customer
- Want to assign taxis to customers in order to minimize the total distance traveled by all taxis (save gas)
  - See code example (taxis.py) for worked solution

What are the variables? What are the constraints?

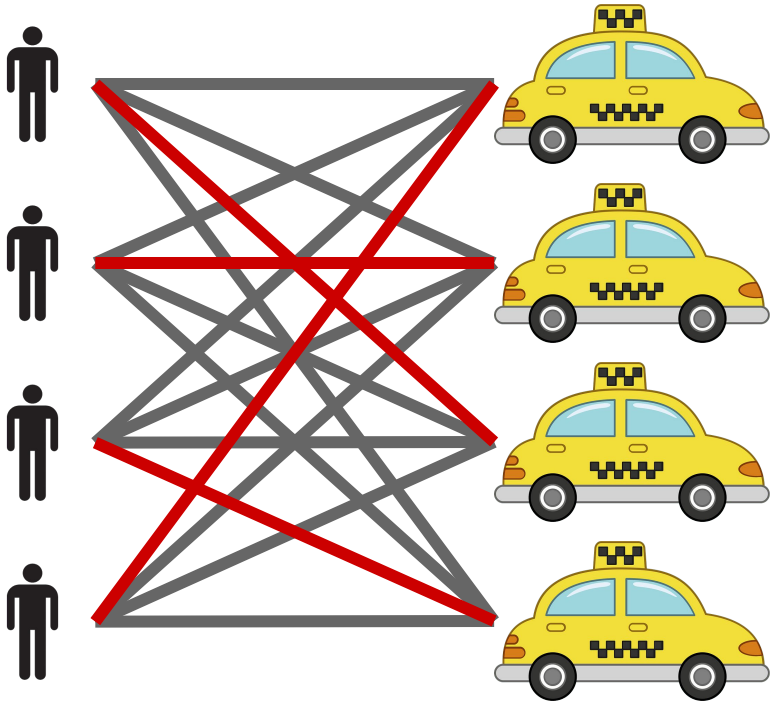
Visualizing The Problem



Visualizing The Problem



Visualizing The Problem



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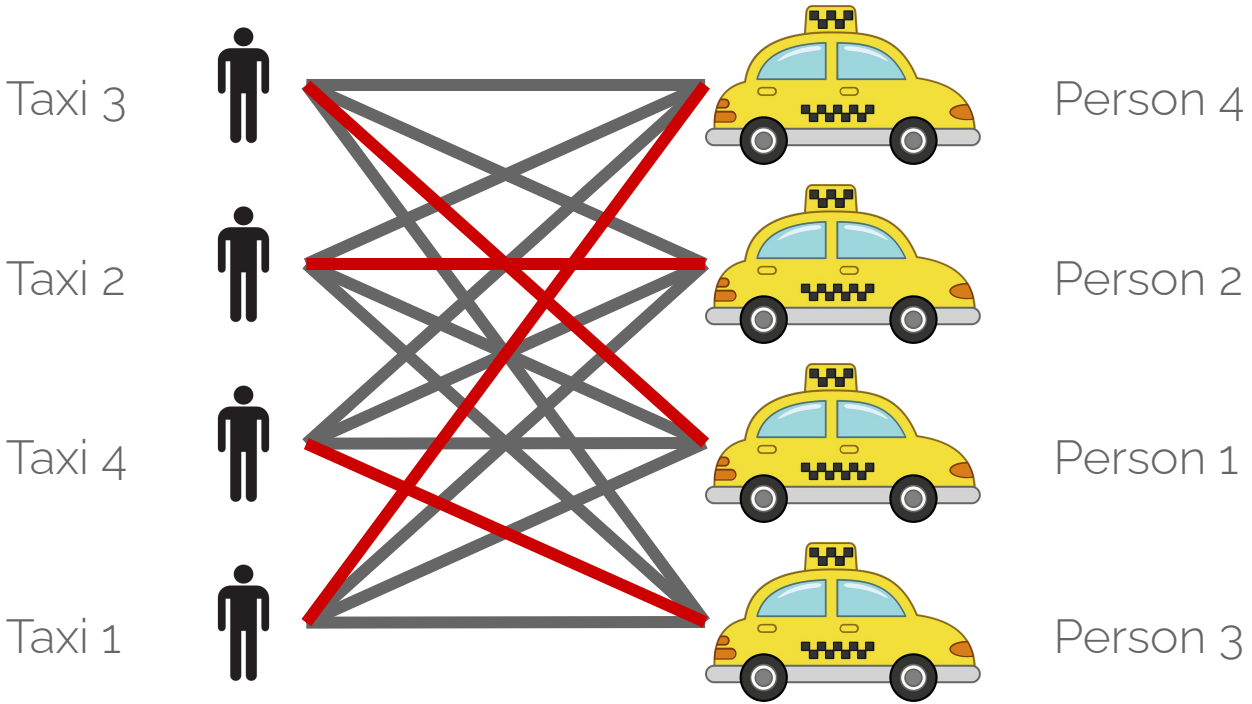
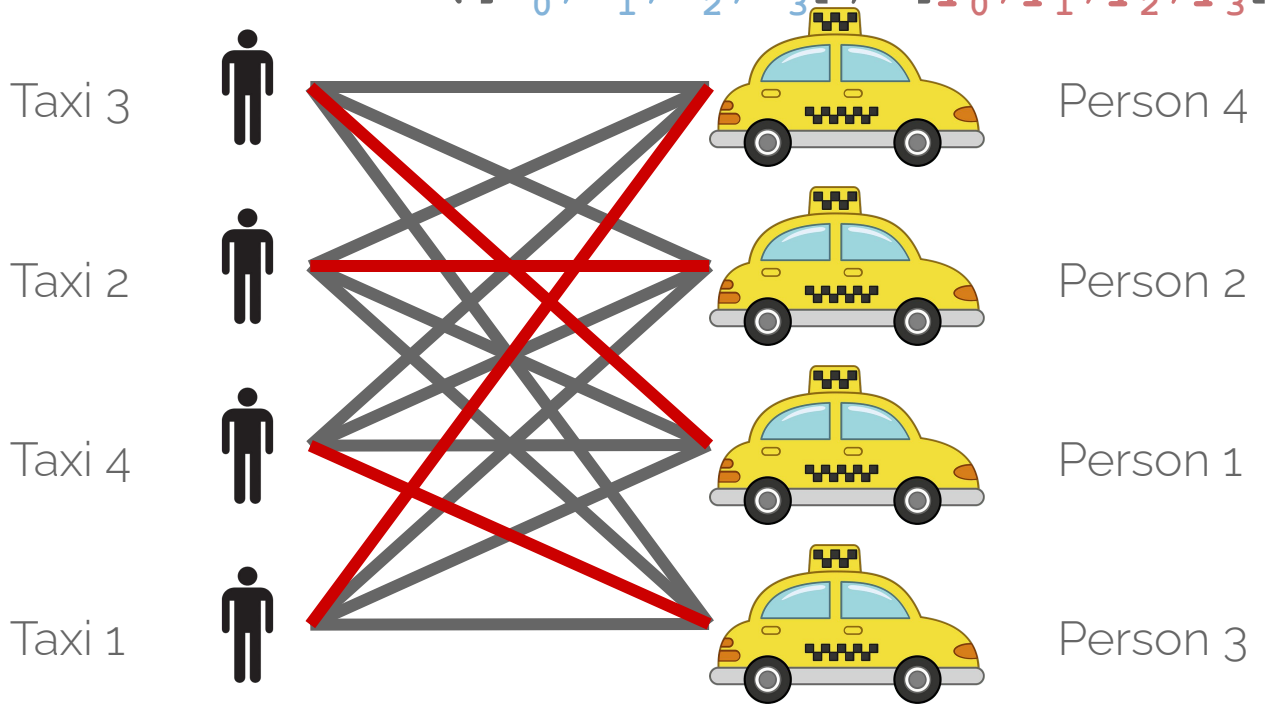


Visualizing The Problem

- `model.AddInverse([x0,x1,x2,x3], [y0,y1,y2,y3])`



`[3,2,4,1]` `[4,2,1,3]`

Interval Variables

- CP-SAT has special variables that provide “syntactic sugar” for representing time intervals
- **`model.NewIntervalVar(start, duration, end, name)`**
- Represents an interval, enforcing `end - start == duration`
  - `start, end, duration` can be constants or variables!
- Note: there is no way to access `start, end, duration` of an interval by default
  - Recommended: directly add them as fields of the interval object

Interval Variables

- Note: there is no way to access `start, end, duration` of an interval by default
  - Recommended: directly add them as fields of the interval, e.g. `interval.start = start`
- **`model.AddNoOverlap(interval_arr)`**
- Powerful constraint: enforces that all intervals in the array do not overlap with each other!
  - It's OK to have shared start/endpoints

Job Shop Scheduling

- m **machines** that do **tasks** which take varying time to finish
  - Machines can do only one task at a time
  - Once a task is started, it must be finished
- n **jobs,** each consisting of a list of tasks
  - Each task must be performed on one specific machine
  - Each task in a job cannot be started until the previous task in the job finished
- **Goal:** minimize the **makespan** (time to finish all jobs)

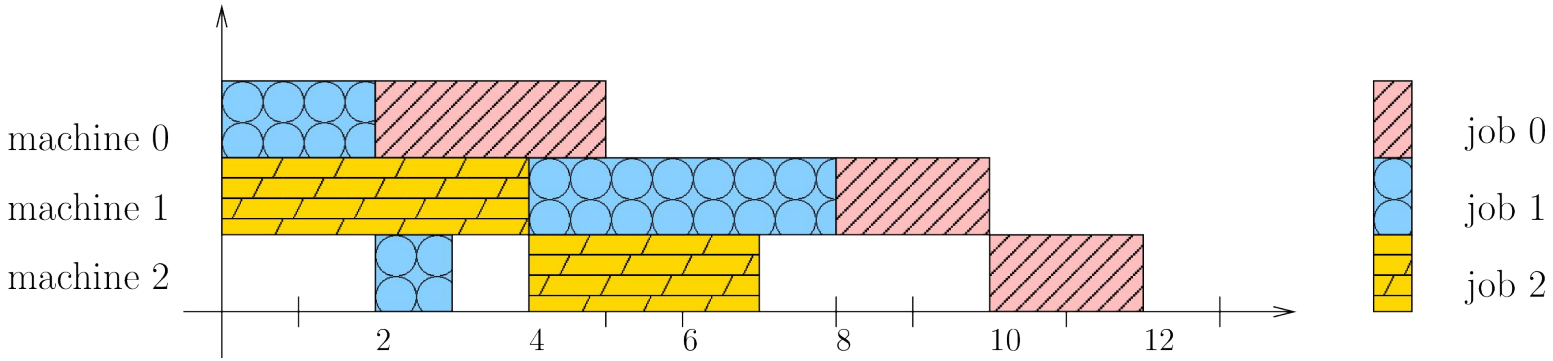
Ex: Job Shop Scheduling

- 3 machines, numbered 0, 1, 2
- Tasks are pairs of (which machine, time required)
- 3 jobs:

```
jobs_data = [  # task = (machine_id, processing_time).
    [(0, 3), (1, 2), (2, 2)],  # Job 0
    [(0, 2), (2, 1), (1, 4)],  # Job 1
    [(1, 4), (2, 3)]           # Job 2
]
```

Ex: Job Shop Scheduling

- Sample feasible (not optimal) solution



- What's the makespan of this solution?
  - See code example (jobshop.py) for worked solution