CIS 1921

Lecture 9: More Constraint Programming

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Logistics

- **HW4** due Monday 11/11
- **Project proposals** due yesterday
  - Check gradescope for feedback when released
- **Project checkpoint** due 11/21
  - Aim for ~75% completion

Recap: Constraint Programs

- Find an assignment of variables to values, subject to general constraints
- Discrete, finitely bounded domains (integers only)
- May or may not optimize an objective

Constraints for BoolVars

- Recall `model.NewBoolVar(name)`
  - Equivalent to `model.NewIntVar(0, 1, name)`
- `boolvar.Not()`
- `model.AddBoolOr(boolvars_list)`
- `model.AddBoolAnd(boolvars_list)`
- `model.AddImplication(b1, b2)`

Ex: Magic Sequence

- A **magic sequence** is a sequence $s_0, s_1, \dots, s_n$ where s_i = number of occurrences of i in the sequence
- Ex:

s_0	s_1	s_2	s_3	s_4
?	?	?	?	?

Ex: Magic Sequence

- A **magic sequence** is a sequence $s_0, s_1, \dots, s_n$ where s_i = number of occurrences of i in the sequence
- Ex:

s_0	s_1	s_2	s_3	s_4
2	1	2	0	0

Reification

- Allows us to express “if-then” relationships as constraints
  - Ex. “If x is equal to 5, then y must be greater than 7”
- **Reification:** the process of linking a logical condition to a boolean variable

- Introduce a new boolean (0/1) variable b which is true if and only if constraint c holds ($b \Leftrightarrow c$)
  - Essentially, name truth value of c with variable b

Reification

"If x is equal to 5, then y must be greater than 7"

- **Step 1:** Introduce a boolean variable which will indicate whether x = 5

```
is_x_five = model.NewBoolVar("is_x_five")
```

- **Step 2:** Tie the boolean indicator with the condition x = 5

$$x == 5 \iff \texttt{is\_x\_five} = 1$$

```
model.Add(x == 5).OnlyEnforceIf(is_x_five)
model.Add(x != 5).OnlyEnforceIf(is_x_five.Not())
```

- **Step 3**: Add further constraints with respect to the indicator:

```
model.add(y > 7).OnlyEnforceIf(is_x_five)
```

Reification in OR-Tools

- OR-Tools API uses **half-reification:** instead of $b \Leftrightarrow c$, just supports $b \Rightarrow c$
  - Can fully reify by combining $b \Rightarrow c$ and $\overline{b} \Rightarrow \overline{c}$

- `constraint.OnlyEnforceIf(bool_var)`
  - Means `bool_var` $\Rightarrow$ `constraint`

⚠️ Reification Warning

- `constraint.OnlyEnforceIf` only works for these constraints:
  - `Add`
  - `AddBoolOr`
  - `AddBoolAnd`
  - `AddLinearExpressionInDomain` (haven't seen this one yet)

- This is usually all you need

Magic Sequence in OR-Tools

- Initialize model and s_i variables

```python
model = cp_model.CpModel()

# Create s_i variables
S = {}
for i in range(n+1):
    S[i] = model.NewIntVar(0, n+1, f's_{i}')
```

Magic Sequence in OR-Tools

- Reify constraints $s_i = j$ into new boolean variables

```
# Reified constraints: eq[i, j] <-> s_i == j
eq = {}
for i in range(n+1):
    for j in range(n+1):
        eq[i, j] = model.NewBoolVar(f's_{i} == {j}')
        model.Add(S[i] == j).OnlyEnforceIf(eq[i, j])
        model.Add(S[i] != j).OnlyEnforceIf(eq[i, j].Not())
```

Magic Sequence in OR-Tools

- Make s_i equal to number of occurrences of i

```python
# s_i = number of occurrences of i in sequence
for i in range(n+1):
    model.Add(
        S[i] == sum(eq[j, i] for j in range(n+1))
    )
```

Magic Sequence in OR-Tools

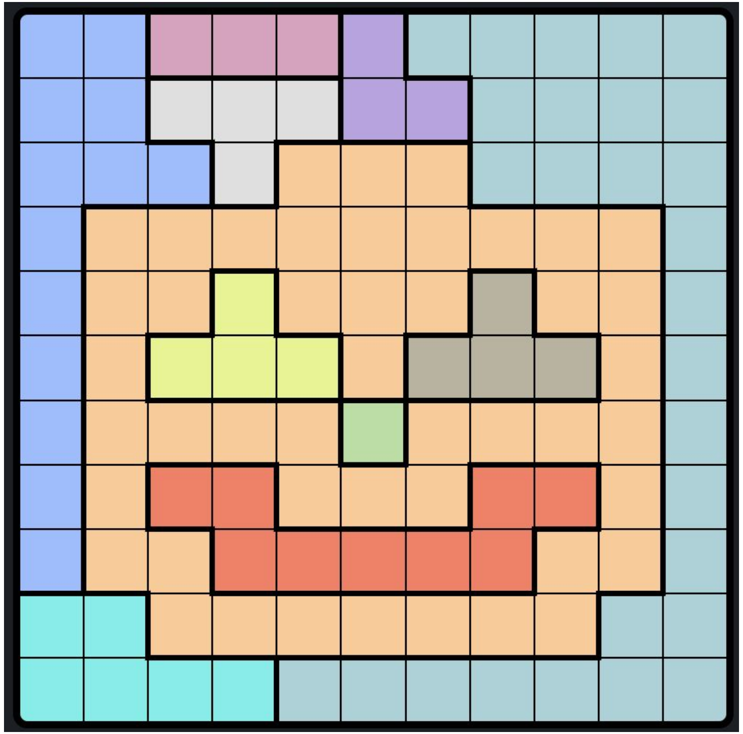
- Solve and print the output

```
solver = cp_model.CpSolver()
if solver.Solve(model) == cp_model.FEASIBLE:
    print([f's_{i}={solver.Value(S[i])}' for i in range(n+1)])
```

```
['s_0=2', 's_1=1', 's_2=2', 's_3=0', 's_4=0']
```

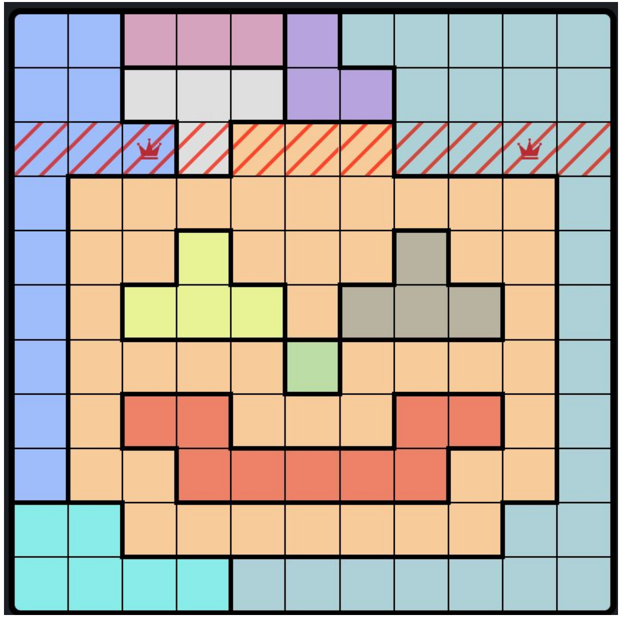
Queens Puzzle

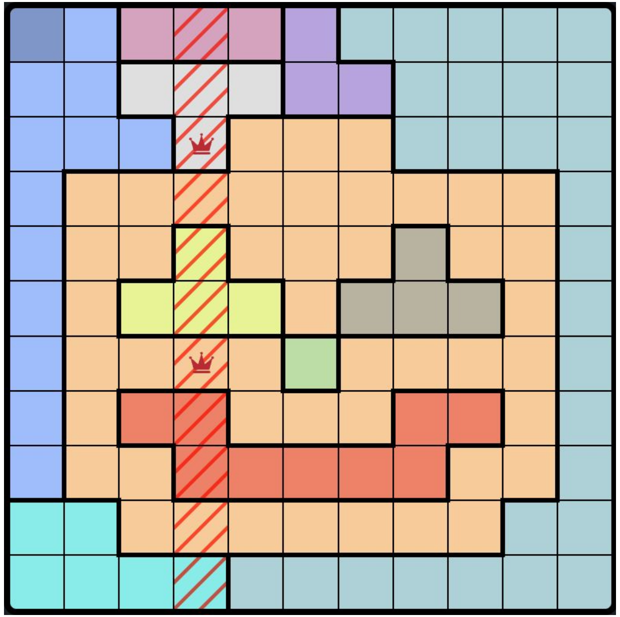
- You are presented with an n x n board, and must place n “queens” in the board



Rules

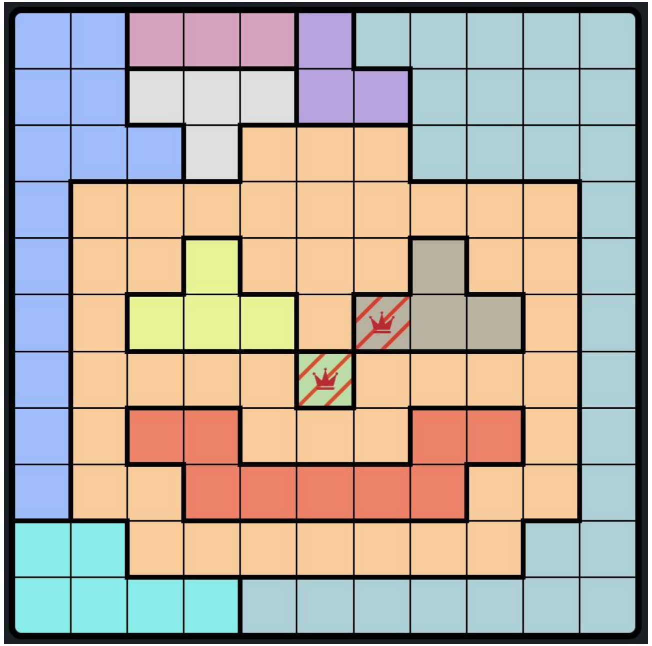
- No two queens can be in the same row or column





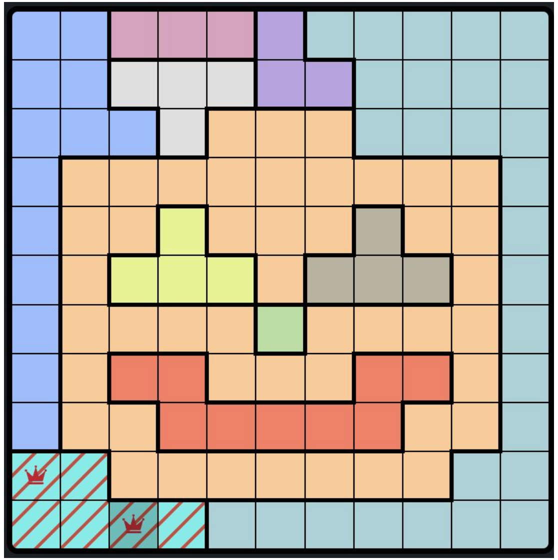
Rules

- No two queens can *touch* diagonally



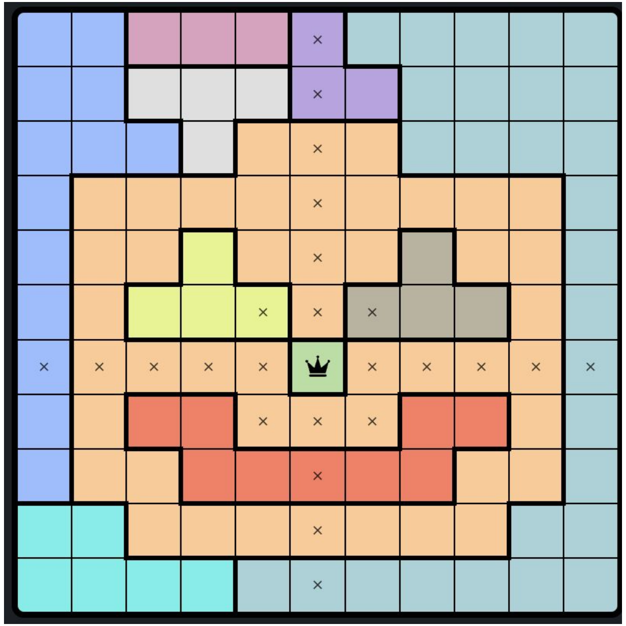
Rules

- No two queens can be in the same region
  - Equivalently, each region must have *exactly* one queen



Observation

- A single queen eliminates the following squares for other queens



Solving The Problem

- **Step 1: Define your variables**

  Remember, the variables are the quantities that *change*, whose values are determined by the solver, and should indicate the solution to your problem.

  Have a variable for the location of each queen. We'll actually do this by maintaining a “row” and “column” variable for each queen

```
rows = [model.NewIntVar(0, BOARD_SIZE - 1, f"row_{i}") for i in range(BOARD_SIZE)]
columns = [model.NewIntVar(0, BOARD_SIZE - 1, f"col_{i}") for i in range(BOARD_SIZE)]
```

Solving The Problem

- **Step 2: Implement your Constraints**

What are the “easiest” constraints in this problem?

- Each queen must be in a different row
- Each queen must be in a different column

Each of these takes just **one line of code** to implement. How?

```
model.AddAllDifferent(rows)
model.AddAllDifferent(columns)
```

Solving The Problem

- **Step 2: Implement your Constraints**

What other constraint is there?

- Queens cannot be one-step diagonally from one another

HACK: Consider the following equality. When is it satisfied?

$$|\texttt{row}[i] - \texttt{row}[j]| + |\texttt{col}[i] - \texttt{col}[j]| = 2$$

Solving The Problem

- **Step 2: Implement your Constraints**

$$|\texttt{row}[i] - \texttt{row}[j]| + |\texttt{col}[i] - \texttt{col}[j]| = 2$$

Satisfied under one of the following conditions:

- Rows are 2 apart, and columns are the same
- Columns are 2 apart, and rows are the same
- Rows are 1 apart and columns are 1 apart

Solving The Problem

- **Step 2: Implement your Constraints**

$$|\texttt{row}[i] - \texttt{row}[j]| + |\texttt{col}[i] - \texttt{col}[j]| = 2$$

Satisfied under one of the following conditions:

- ~~Rows are 2 apart, and columns are the same~~
- ~~Columns are 2 apart, and rows are the same~~
- Rows are 1 apart and columns are 1 apart

Solving The Problem

- **Step 2: Implement your Constraints**

$$|\texttt{row}[i] - \texttt{row}[j]| + |\texttt{col}[i] - \texttt{col}[j]| = 2$$

Satisfied under one of the following conditions:

- ~~Rows are 2 apart, and columns are the same~~
- ~~Columns are 2 apart, and rows are the same~~
- Rows are 1 apart and columns are 1 apart

Touching diagonally!

Solving The Problem

- **Step 2: Implement your Constraints**

  So we enforce the following:

$$|\texttt{row}[i] - \texttt{row}[j]| + |\texttt{col}[i] - \texttt{col}[j]| \neq 2$$

```python
for i in range(BOARD_SIZE):
    for j in range(i + 1, BOARD_SIZE):
        # Define variables for absolute differences
        abs_row_diff = model.NewIntVar(0, BOARD_SIZE - 1, f"abs_row_diff_{i}_{j}")
        abs_col_diff = model.NewIntVar(0, BOARD_SIZE - 1, f"abs_col_diff_{i}_{j}")

        # Set up absolute difference constraints
        model.AddAbsEquality(abs_row_diff, rows[i] - rows[j])
        model.AddAbsEquality(abs_col_diff, columns[i] - columns[j])

        # Ensure that the queens are not exactly one cell away diagonally
        model.Add(abs_row_diff + abs_col_diff != 2)
```

Solving The Problem

- **Step 2: Implement your Constraints**

  What other constraint is there?

  - Queens cannot be in the same region

This is inherently different from ensuring the queens are in different rows and columns

- Rows and Columns were easy, because our variables were defined with respect to rows and columns
- Here, the regions are strange shapes, and isn't as easy as ensuring row[i] != row[j]

Solving The Problem

- **Step 2: Implement your Constraints**
  - Queens cannot be in the same region

For each cell in a region, maintain an indicator, for if a queen is present in that cell. Then, group all cells of a region together, and ensure that **the sum** of the indicators for these cells is equal to... 1

Is reification necessary? **YES!**

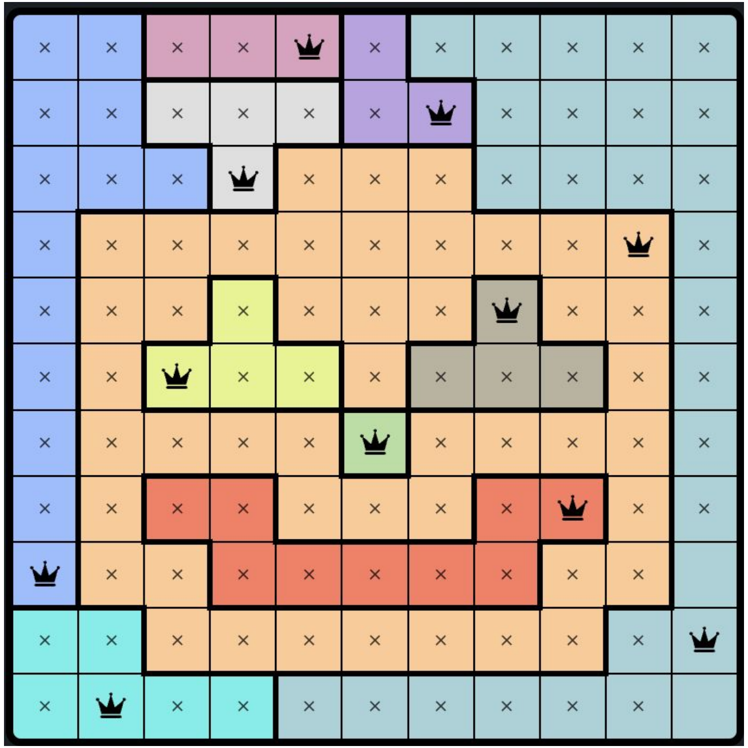
A boolean variable on its own doesn't tell our solver about the position of our queen. We need to **link** this boolean variable to our queen variables.

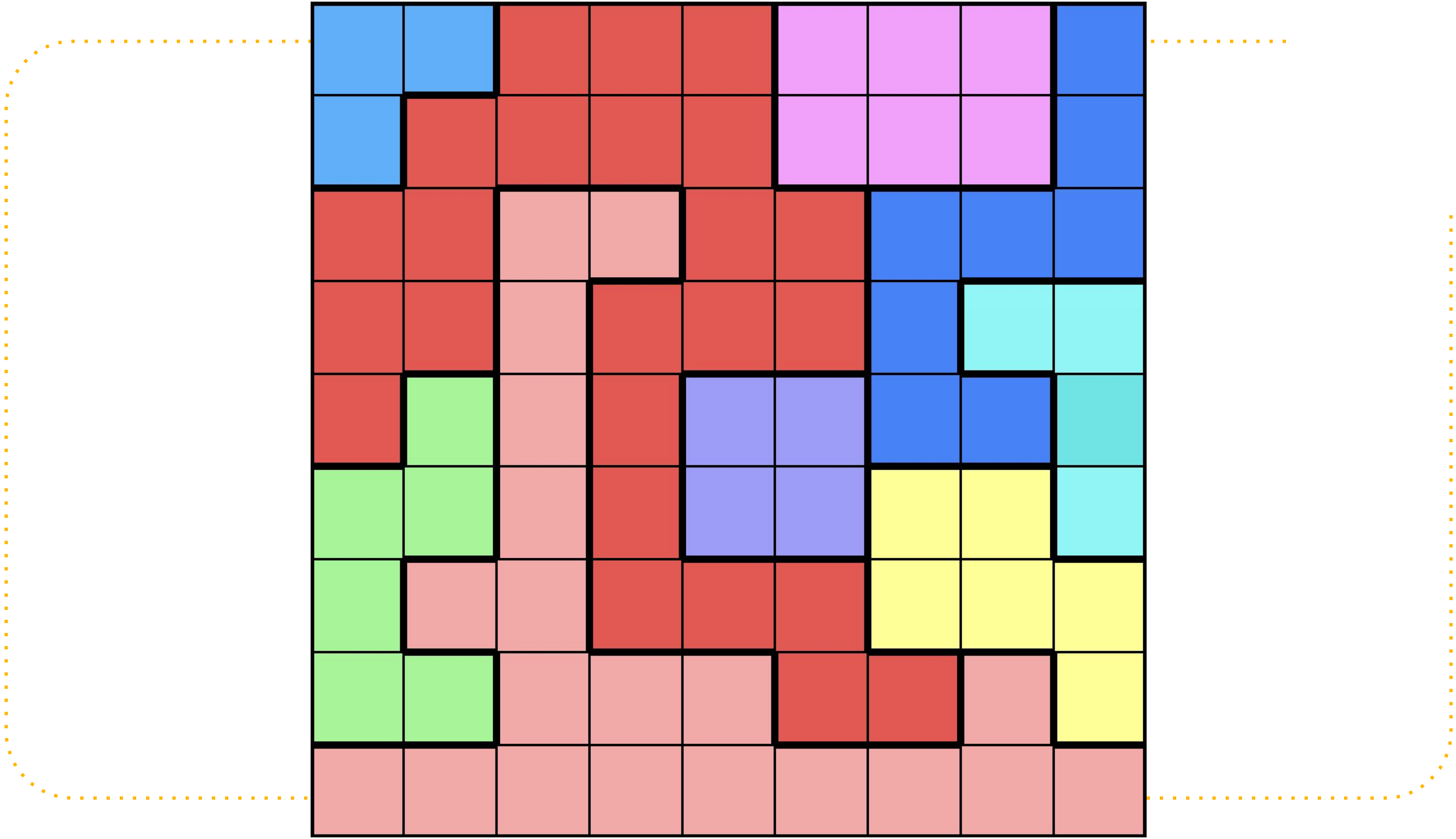
Solving The Problem

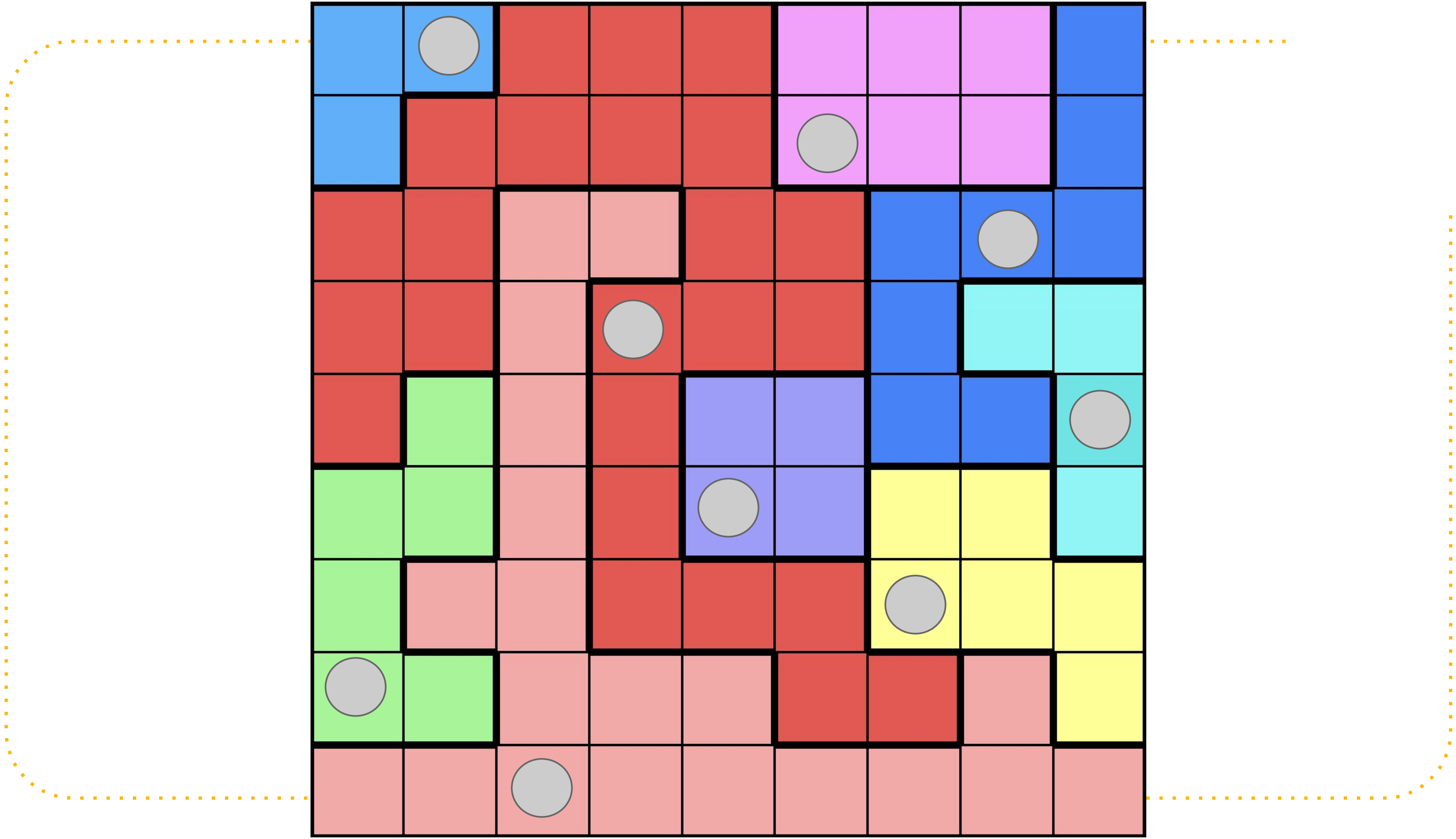
- **Step 2: Implement your Constraints**
  - Queens cannot be in the same region

```
# Enforce region constraint: exactly one queen in each region
for region_number, region in enumerate(regions):
    in_region = []
    for (row, col) in region:
        cell_var = model.NewBoolVar(f"cell_{row}_{col}")
        in_region.append(cell_var)
        model.Add(rows[region_number] == row).OnlyEnforceIf(cell_var)
        model.Add(columns[region_number] == col).OnlyEnforceIf(cell_var)
    model.Add(sum(in_region) == 1)
```

Solution







Non-contiguous Domains

- `cp_model.Domain.FromValues([0,2,4,6,8])`

0	1	2	3	4	5	6	7	8

- `cp_model.Domain.FromIntervals([0, 2],[6, 8])`

0	1	2	3	4	5	6	7	8

- `model.NewIntVarFromDomain(domain, name)`

Linear Expressions on Domains

- Enforce that result of a linear expression must fall into a domain
- ```
  cp_model.AddLinearExpressionInDomain(
      x + y,
      cp_model.Domain.FromValues([0,2,4])
  )
  ```

0,0	1,0	2,0	3,0	4,0
0,1	1,1	2,1	3,1	4,1
0,2	1,2	2,2	3,2	4,2
0,3	1,3	2,3	3,3	4,3
0,4	1,4	2,4	3,4	4,4
  ```

Ex: Shipping Allotments

- Shipping company has n ships with capacity 100 each
- Want to load all shipments of varying sizes onto ships
- **Goal:** maximize number of ships which have at least 20 capacity unused (in case of emergency)
  - See worked solution in additional code (ships.py)

Tuning the CP-SAT Solver

- We can play around with CP-SAT internals to possibly speed up the search
- There are tons of parameters that can be adjusted
  - Some are documented better than others...
  - https://github.com/google/or-tools/blob/stable/ortools/sat/sat_parameters.proto
- **Warning:** these things are generally far less important than having a good encoding

Parallelization

- We can run solver computation in parallel across multiple threads

```
solver = cp_model.CpSolver()
solver.parameters.num_search_workers = 4
```

  - By default, CP-SAT will try to use all available cores

Hinting

- We can give the model a **hint** to try setting a variable to a specified value

```
# try setting x = 5 first
model.AddHint(x, 5)
```

Quick & Dirty Optimization

- Finding an optimal solution can take far longer than finding a feasible solution
- Often in practice, we don't *really* care about having the true optimal value with total certainty
  - Just want it to be "close enough"

Quick & Dirty Optimization

Solution:

- Optimize objective and run solver for a reasonable amount of time (depends on your patience)
- Interrupt early with Ctrl+C or `max_time_in_seconds` param
    - If interrupted, solver returns FEASIBLE instead of OPTIMAL
- Print the intermediate objective value and solution and decide if it's "good enough"
    - For tough problems, no guarantee that you are close to optimal!
    - `best_bound` in response stats gives best LB (when minimizing) or UB (when maximizing) proved so far for optimal value

Quick & Dirty Optimization

- Helpful: set `log_search_progress` param to True
  - Prints every time a new best solution is found
- Sometimes helpful: custom solution callback
  - Called each time any new feasible solution is found

```python
class BestSolutionFinder(cp_model.CpSolverSolutionCallback):

    def __init__(self, minimizing=True):
        cp_model.CpSolverSolutionCallback.__init__(self)
        self.minimizing = minimizing
        self.best_value = (1 if minimizing else -1) * float('inf')

    def on_solution_callback(self):
        obj = self.ObjectiveValue()
        if (self.minimizing and obj < self.best_value) \
        or (not self.minimizing and obj > self.best_value):
            self.best_value = self.ObjectiveValue()
            print(f'New best value: {self.best_value}')
```

```python
solver = cp_model.CpSolver()
solver.parameters.num_search_workers = 6
solver.parameters.log_search_progress = True
# Our solution callback is redundant to logging
best = BestSolutionFinder()
solver.SolveWithSolutionCallback(model, best)
```

Approximating Feasibility

- What if non-optimization problem is too hard to solve?
- Can't interrupt early for a "good enough" solution; intermediate solution is feasible or it is not
- What if we were OK with a "not quite feasible" solution?
  - What could "not quite feasible" mean?

Soft Constraints

- Constraints like `Add(...)` are **hard constraints**
  - Must be satisfied
- **Soft constraints**: can be violated, but incurs a penalty
- Transform feasibility problem into optimization problem by minimizing penalty
  - Allows interrupting early if you're OK with some violated constraints
  - Can sometimes be faster than solving with hard constraints!

Ex: Soft Graph Coloring

- Hard constraint:

  for every edge (u, v), $color(u) \neq color(v)$

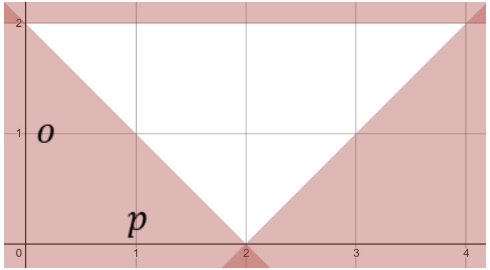
- Soft constraint

  $penalty$ = num. edges (u, v) with $color(u) = color(v)$

- Can count number of violated constraints using reification

Optimizing Pairs of Objectives

- What if we want to add soft constraint with penalty p but problem already optimizes (say, minimizes) objective o?
- **Key idea:** why not minimize both?
- Attempt 1: minimize $\boldsymbol{o + p}$
  - **Problem:** o and p may be interrelated
  - E.g., minimum possible value of o may be lower when $p = 1$ than when $p = 0$



Optimizing Pairs of Objectives

- **Observation**: avoid interdependence by minimizing p first and using o to break ties
  - Aka, minimize (p, o) over the **lexicographic ordering**
- How to make sure that p is minimized before o?
- Attempt 2:
  - minimize $\boldsymbol{Mp} + \boldsymbol{o}$, where $M = o_{max} - o_{min} + 1$
- Can generalize to maximization

Optimizing Pairs of Objectives

- Previous approach doesn't scale well for >2 objectives
- What's another way to do it using multiple calls to `Solve`?

```
model.Minimize(p)
solver.Solve(model)

# Hint (may speed up solving)
model.AddHint(p, solver.Value(p))
model.AddHint(o, solver.Value(o))

# Minimize o (and constrain p based on previous optimal value)
model.Add(p == solver.Value(p)) # use >= or <= if not optimal
model.Minimize(o)
```

General CP-SAT Modeling Tips

- Don't be afraid to add new variables/constraints, but be aware of roughly how many you have ($O(n)$? $O(n^3)$?)
- Try to restrict range of values for each variable
- Use boolean variables/constraints when possible
- Experiment with hard vs. soft constraints
- If possible, split into subproblems, then combine solutions
- Make it easy to toggle constraints on/off for debugging

MIP vs CP-SAT

MIP	CP-SAT
• Supports infinite bounds • Supports fractional variables and coefficients • Better handles LP-style problems (with integers mixed in) • Reification of constraints is possible, but requires algebraic modeling trick	• Better handles combinatorial problems, Booleans • More sophisticated interface • Lots of specialized modeling objects • Modeling may be easier • Models may be more extensible • Reification is easier, more performant

- Neither is clearly more performant in general
- Neither is an evolution of the other

- Happy Halloween!
- Happy Diwali!
- Don't forget to vote!