CIS1921

Lecture 2: Solvers & Encoding

Reminders

- Homework 0 due Monday, Sept 9, 11:59PM
- Cindy's OH: Wed 6-7pm
- Ishaan's OH: Thurs 6-7pm (subject to change, will announce on Ed)
- OH in Levine 3rd floor bump space (for now)
- Homework 1 will be posted either after class or sometime in the next day or two. Will have 2 weeks to complete it.
- Make sure you have access to Ed
- Gradescope: **3RDWG3**

Accommodations

- If you have any special accommodations (i.e. things that would prevent you from submitting a HW at a certain time), please let me know ASAP.

A couple past final projects...

Constrained Style Sheets

Kaan Erdogmus & Shriyash Upadhyay

```
c(p == min(h, w))
c(l * 10 == p)
c(2l == l * 2)

box-v(p)-neuv(inset) {
  box-shadow:
      v(inset) v(l)px v(l)px
      v(2l)px #bebebe,
      v(inset) -v(l)px -v(l)px
      v(2l)px #ffffff;
}
```

Optimal Asset Portfolios

Soham Dharmadhikary & Nikhil Kokra

An Optimization Problem

maximize return, minimize covariance and shorting cost

Maximize:

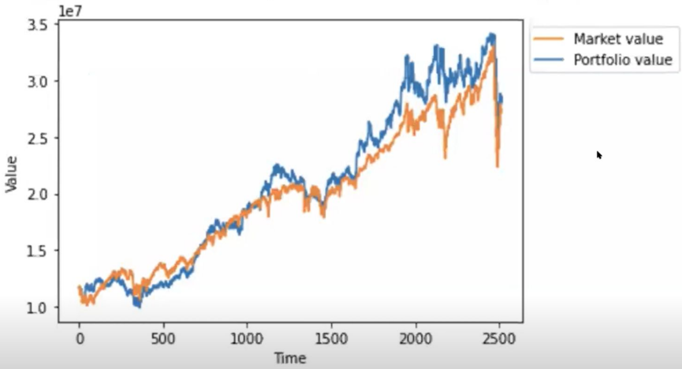
$$(V \cdot \vec{r})^T \vec{w} - A||\vec{w}V|| - \sum_{i=1}^{M} k * 1(w_i < 0)$$

Subject To:

$$\sum_{i=1}^{M} w_i = 1$$

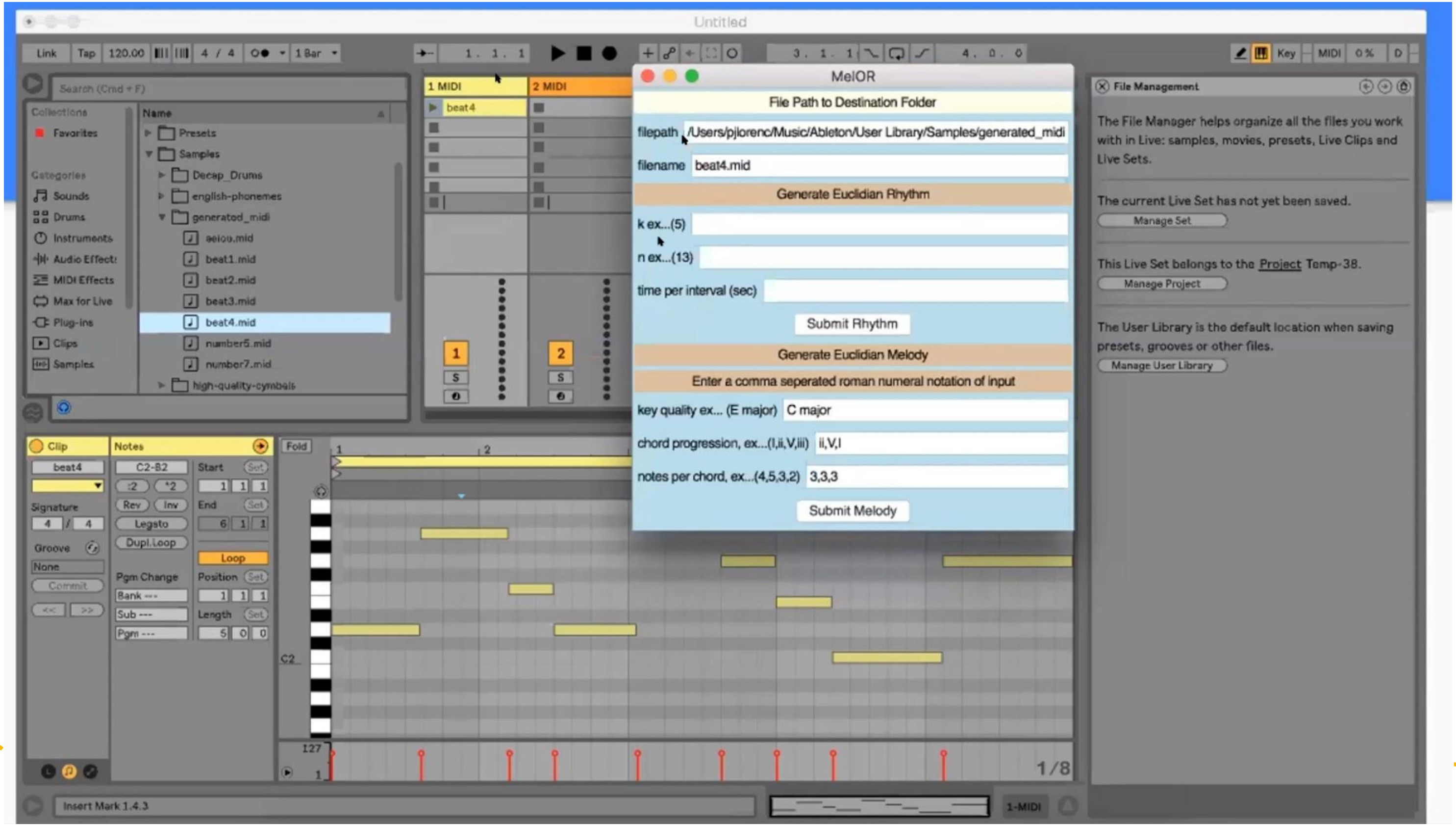
Collecting and Evaluating Results

```
Portfolio return (2010 - 2020): 2.3963905327367754
Portfolio variance (2010 - 2020): 0.011153992575588882
Market return (2010 - 2020): 2.3295003556295586
Market variance (2010 - 2020): 0.010533059982624654
```



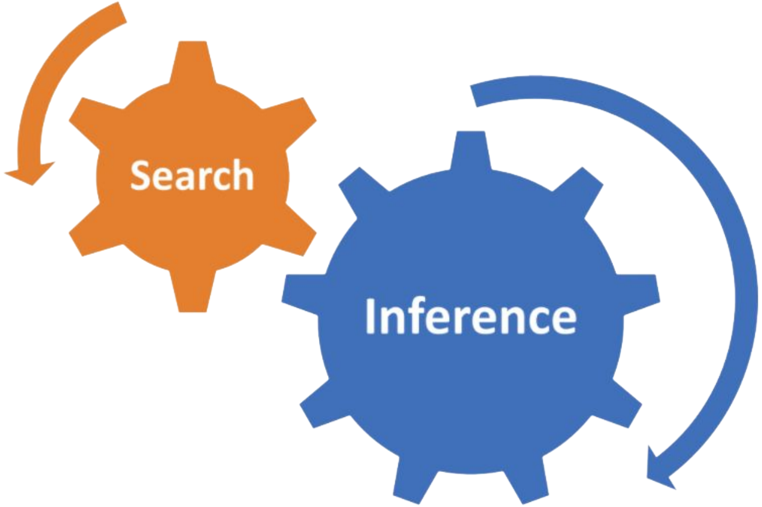
Generative Melody Creator

Paul Lorenc & Leonardo Nerone

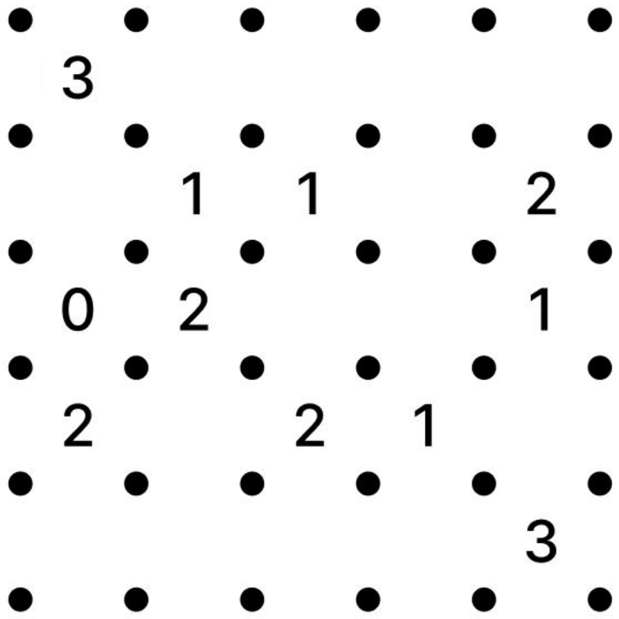


Let's play a game

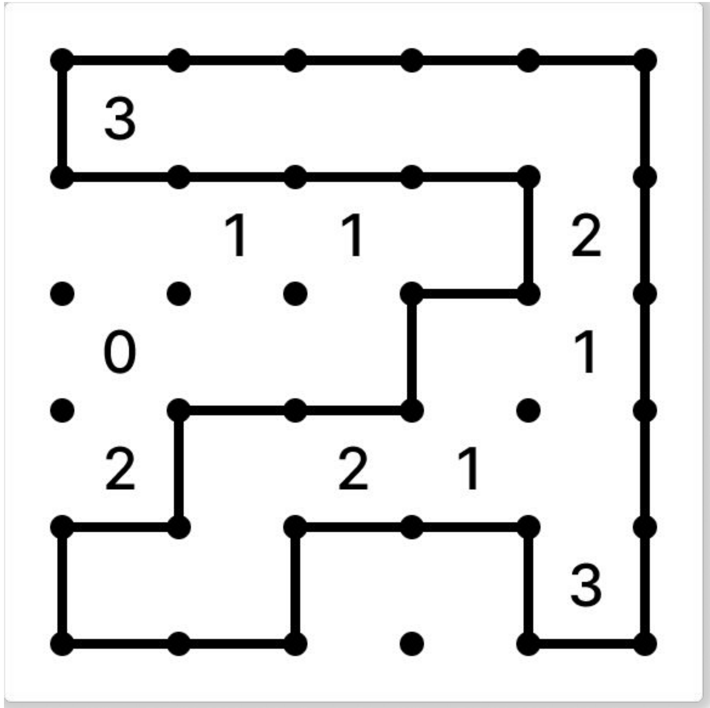
Search & Inference



Problem



Solution



Recap

Last week

- Intro to hard problems
- The SAT problem

This week

- Using SAT solvers in Python
- Factors that affect solver runtimes

Recall: SAT Problem

Given a formula φ of boolean variables, does there exist a truth assignment that makes the entire formula evaluate to True?

- Ex: $(x \vee y) \Rightarrow y$ is satisfiable with $\{x = T; y = T\}$
- Ex: $(x \wedge \overline{x})$ is unsatisfiable

SAT terminology

- Assume only logical symbols are AND, OR, NOT
- **Literal:** a boolean variable (x) or its negation ($\overline{x}$)
  - (x) is called a **positive** literal, and ($\overline{x}$) is a **negative** literal
  - “a variable as it appears in a formula”
- **Clause:** a disjunction/OR of literals
  - e.g. $(\overline{x} \vee y \vee z)$

- Note: we would say that $(\overline{x} \vee y) \wedge (x \vee y)$ has 2 variables and 4 literals

Conjunctive Normal Form

- A boolean formula is in **conjunctive normal form (CNF)** if it is a conjunction/AND of clauses (i.e., an AND of ORs)
  - "a CNF" means "a formula in CNF"
- Ex: which of the following are in CNF?
  - $(\overline{x} \vee y \vee z) \wedge (x \Rightarrow w)$
  - $(\overline{x} \wedge y \wedge z) \vee (\overline{y} \wedge z)$
  - $(\overline{x} \vee y \vee z) \wedge (\overline{y} \vee z)$
  - $\overline{x} \vee y \vee z$
  - $x \wedge \overline{x}$

CNF-SAT: a loss of generality?

- It's convenient for SAT solvers to accept formulas in CNF, but what if we need to solve any other non-CNF boolean formula?
- **Every SAT problem can be converted to a CNF-SAT problem**
  - New problem will only be linearly bigger
  - If curious, google the Tseitin transformation

Notation

- Positive (x) and negative ($\overline{x}$) literals
- Recall that we often consider a clause as a **set of literals**, and a CNF as a **set of clauses**
- Therefore might express CNFs in **compact notation**:

$$\{x\overline{z}, yz\overline{x}\} \quad \simeq \quad (x \vee \overline{z}) \wedge (y \vee z \vee \overline{x})$$

MiniSAT

- **MiniSat** is a minimal, open source SAT solver created by Niklas Eén & Niklas Sörensson
  - *An Extensible SAT-solver* (Eén & Sörensson, 2003)
- Silver medal in SAT Comp. 2005...
- ...with only ~600 lines of C++ code!

Niklas Eén

Niklas Sörensson

PicoSAT

- **PicoSAT** is an open source SAT solver created by Prof. Armin Biere
    - *(Q)CompSAT and (Q)PicoSAT at the SAT'06 Race (Biere 2006)*
- Armin Biere: *Handbook of Satisfiability*
- Gold medalist in SAT Competition 2007
- Written in C, but bindings to Python, JS, R, etc.
    - We'll use this: can be easily installed with pip!

Armin Biere

DEMO: Solving a CNF with PicoSAT

Let's solve the following formula:

$$\varphi = \{5, \overline{4}1, \overline{4}2, 4\overline{1}\overline{2}, \overline{5}43, 5\overline{4}, 5\overline{3}\}$$

```
cnf = [[5],[-4,1],[-4,2],[4,-1,-2],[-5,4,3],[5,-4],[5,-3]]
pycosat.solve(cnf) # That was easy!
```

```
[1, 2, -3, 4, 5]
```

SAT Encodings

Why do we care about SAT?

Can encode most problems as an instance of CNF-SAT. Let's see how!

Warmup: Boolean Encodings

$x \Rightarrow y$	$\{\overline{x}y\}$
$x \Leftrightarrow y$	$\{\overline{x}y, \overline{y}x\}$
$x \oplus y$	$\{xy, \overline{xy}\}$
if x then y else z	$\{\overline{x}y, xz\}$

First-Order Logic Encodings

$All(x_1, \ldots, x_n)$	$\{x_1, x_2, \ldots, x_n\}$
$ALO(x_1, \ldots, x_n)$	$\{x_1 x_2 \cdots x_n\}$
$AMO(x_1, \ldots, x_n)$	$\{\overline{x}_i \overline{x}_j \mid 0 \leq i < j \leq n\}$
$ExactlyOne(x_1, \ldots, x_n)$	$ALO(\ldots) \land AMO(\ldots)$

Complexity of Encoding

$AMO(x_1, \ldots, x_n)$ $\{\overline{x}_i\overline{x}_j \mid 0 \leq i < j \leq n\}$

- Called the binomial encoding
- How many clauses in this encoding?
- Other encodings that use fewer clauses, but introduce extra variables

Binary AMO Encoding

$AMO(x_1, \dots, x_n)$

- Retain original variables $x_1, \dots, x_n$
- Let $m = \lceil \lg n \rceil$, and introduce new variables $b_1, \dots, b_m$
  - b_j represents j^{th} bit of T, where x_T is the ($\leq$) one True var
- Clauses:

$$\left\{ \left(x_i \Rightarrow \begin{cases} b_j & \text{if } j^{th} \text{ bit of } i \text{ is } 1 \\ \overline{b_j} & \text{if } j^{th} \text{ bit of } i \text{ is } 0 \end{cases} \right) \middle| \; \forall x_i, b_j \right\}$$

- How many extra variables? How many clauses?

Good Encoding Matters

- The performance of a SAT solver can vary **hugely** depending on the encoding.
- Different encodings work better in different cases: **no "universal best encoding."**
- **Start simple** and be more clever if necessary.

Encoding Graph Coloring

Can we color $G = (V, E)$ with $\leq k$ colors?

Encoding Graph Coloring

Can we color $G = (V, E)$ with $\leq k$ colors?

- x_{ic}: if we assign v_i color c
- **Constraint 1:** every vertex has ≥ 1 color

$$\{x_{i1} x_{i2} \dots x_{ik} \mid \forall\, v_i \in V\}$$

Encoding Graph Coloring

Can we color $G = (V, E)$ with $\leq k$ colors?

- x_{ic}: if we assign v_i color c
- **C2:** if $(u, v) \in E$, then u, v have different colors

$$\{\overline{x_{ic} x_{jc}} \mid \forall (v_i, v_j) \in E, c \in [1..k]\}$$

DEMO Part 1

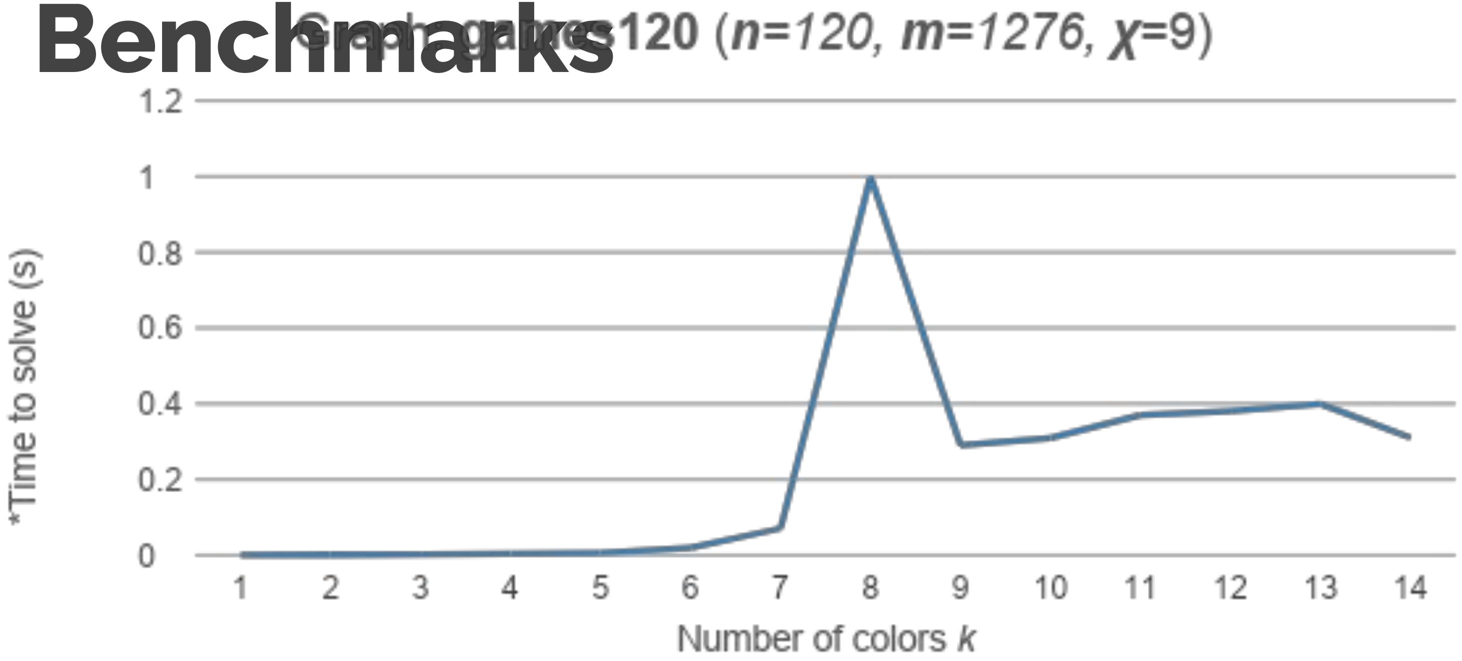
From Coloring to Min-Coloring

- What if we wanted to find the $\chi(G)$, the minimum number of colors to color G?
- Binary search
  - Check if G can be colored with 1, 2, 4, 8, 16, ... colors
  - Stop at smallest value 2^k such that G is 2^k-colorable
  - Binary search for $\chi(G)$ in range $[2^{k-1}, 2^k]$
- Requires $O(\lg \chi(G))$ runs of solver

From Coloring to Min-Coloring

- UNSAT: Given an integer k, is it that there is NO VALID k-coloring?
  - UNSAT can be **way** harder than SAT.
  - Finding just one satisfying assignment (SAT) vs. showing that none exists (massive search space UNSAT)

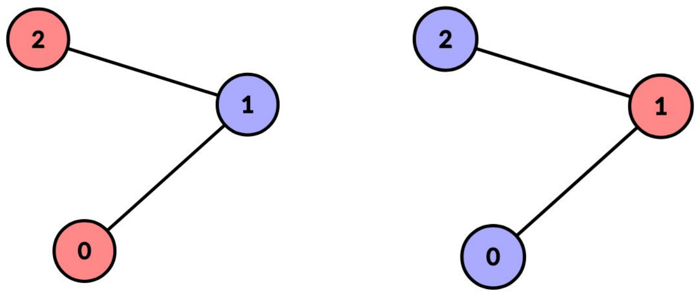
PicoSAT Coloring Benchmarks



*Running on a Dell XPS laptop with 16GB of RAM, in a Jupyter notebook.

Symmetry Breaking

- Solving UNSAT graph coloring problems takes a very long time… why?
- Must rule out every symmetric coloring
- Ex: equivalent colorings

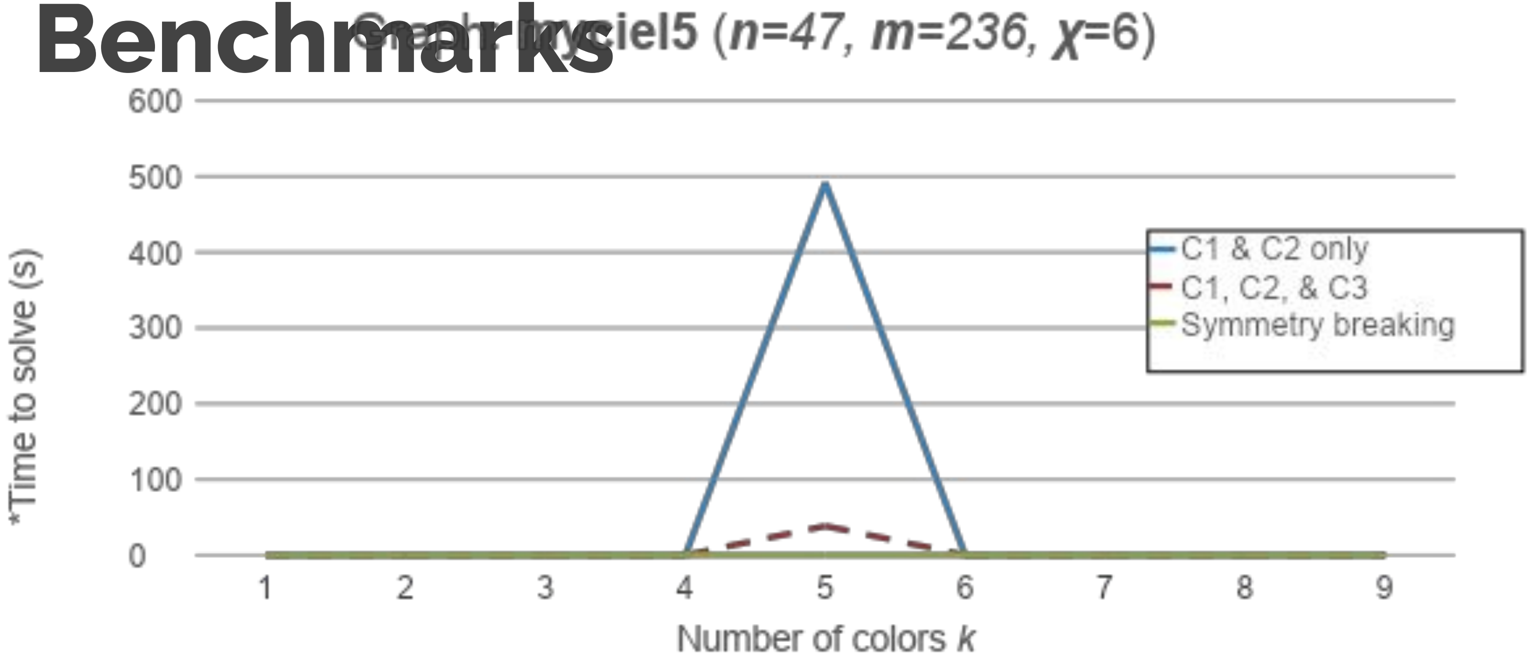


Symmetry Breaking

- Key idea: add constraints that rule out equivalent symmetric colorings
- Basic way to do this: pick some vertices (ideally a dense subgraph) and fix their colors

DEMO Part 2

PicoSAT Coloring Benchmarks



*Running on a Dell XPS laptop with 16GB of RAM, in a Jupyter notebook.

Encoding Stable Matchings

We have n men and n women. Each man and woman submits a preference list ranking everyone of the opposite sex (descending).

Goal: find a **matching** of men to women.

A man and woman who both prefer each other to their matched partners are a **blocking pair**.

A matching is **stable** if it has no blocking pairs.

Encoding Stable Matchings

m_{ip}: if man i is matched to p^{th} woman or later on his list

w_{ip}: if woman i is matched to p^{th} man or later on her list

$[W_1 > W_2]\ M_1 \longleftrightarrow W_1\ [M_1 > M_2]$

$[W_1 > W_2]\ M_2 \longleftrightarrow W_2\ [M_1 > M_2]$

$m_{1,1}$	$m_{1,2}$	$m_{2,1}$	$m_{2,2}$	$w_{1,1}$	$w_{1,2}$	$w_{2,1}$	$w_{2,2}$
T	F	T	T	T	F	T	T

Encoding Stable Matchings

m_{ip}: if man i is matched to p^{th} woman or later on his list

w_{ip}: if woman i is matched to p^{th} man or later on her list

- **C1:** every man is matched

$$\{m_{i1} \mid 1 \leq i \leq n\}$$

(plus symmetric constraints for women for this and the following constraints)

Encoding Stable Matchings

m_{ip}: if man i is matched to p^{th} woman or later on his list

w_{ip}: if woman i is matched to p^{th} man or later on her list

- **C2:** if a man gets his p^{th} or later choice, it's also his $(p-1)^{th}$ or later choice

$$\{m_{ip} \Rightarrow m_{i(p-1)} \mid 1 \leq i \leq n, 2 \leq p \leq n\}$$

Encoding Stable Matchings

m_{ip}: if man i is matched to p^{th} woman or later on his list

w_{ip}: if woman i is matched to p^{th} man or later on her list

- **C3:** if man i is matched to woman j, then she is matched to him also

$$\{m_{ip} \land \overline{m_{i(p+1)}} \Rightarrow w_{jq} \land \overline{w_{j(q+1)}} \mid 1 \leq i,j \leq n\}$$

- $p =$ position of woman j in man i's list
- $q =$ position of man i in woman j's list

Encoding Stable Matchings

m_{ip}: if man i is matched to p^{th} woman or later on his list

w_{ip}: if woman i is matched to p^{th} man or later on her list

- **C4:** if man i is matched to someone worse than woman j, her match must be better than him

$$\{m_{i(p+1)} \Rightarrow \overline{w_{jq}} \mid 1 \leq i, j \leq n\}$$

- p = position of woman j in man i's list
- q = position of man i in woman j's list

Why Stable Matchings?

- Gale-Shapley algorithm solves SM problem in linear time. Why use SAT solvers?
- SMTI: stable matching problem where preference lists may be incomplete and contain ties
- SM-C: stable matching problem with couples
- Our encoding easily generalizes to SMTI, SM-C
- Theorem: SMTI and SM-C are NP-complete.

Next Week

- Start learning how SAT solvers work
- Homework 1 due in two weeks
  - Encoding Sudoku into SAT
  - Extra credit challenge!

Stay Healthy

"For a thinking person, the most serious mental illness is not being sure of who you are." ~Benoit Mandelbrot

References

A. Biere, *Handbook of satisfiability*. Amsterdam: IOS Press, 2009.

J. Burkardt, “CNF Files,” *John Burkardt's Home Page*. [Online]. Available: https://people.sc.fsu.edu/~jburkardt/data/cnf/cnf.html.