CIS 189

Lecture 11: TSP Techniques

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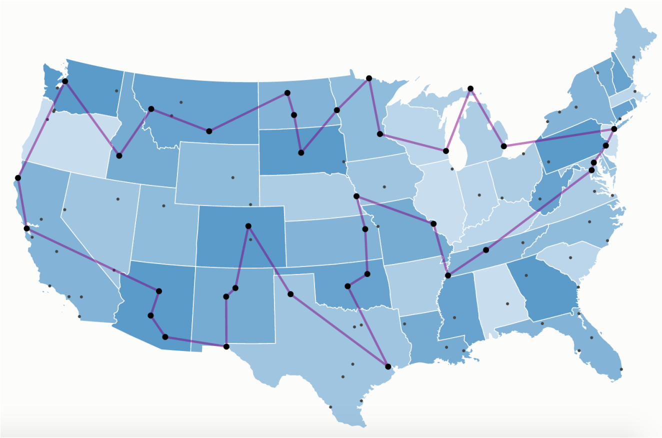
The rest of the semester...

- Finished with our satisfiability + constraint programming saga!
- 3 more lectures on "special topics" + final presentations
- Don't forget: project check-in due 4/18 at 4pm
    - You should be at least 2/3 done with project!

Traveling Salesman Problem

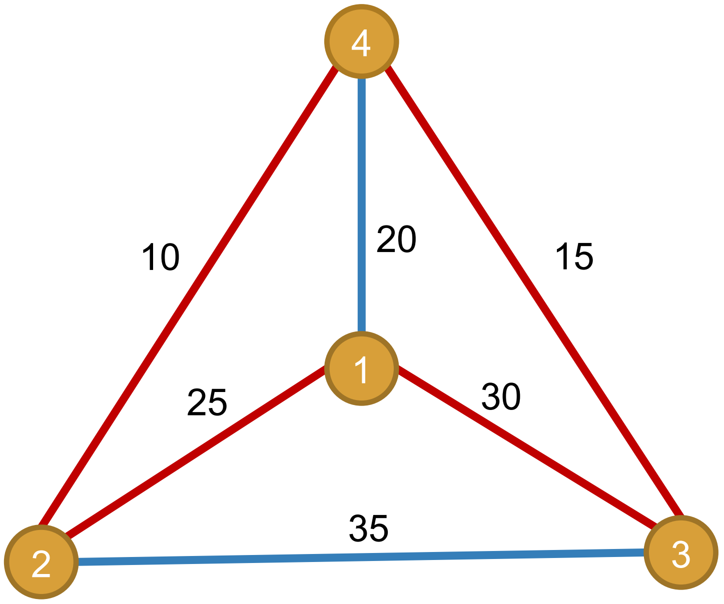
- **Problem:** in weighted complete graph, find a tour of minimum total cost that visits every vertex exactly once and returns to starting vertex
  - Graph can be directed or undirected
- Applications in routing, logistics, producing microchips
- NP-complete!



Preliminary Notation

- We'll look at complete directed graphs (parallel edges, but no self-loops) with n nodes, m edges
  - Undirected graphs are often a special case
- Directed edge $(i, j) = i \rightarrow j$ has weight $w(i, j)$
- We'll denote a **tour** as a permutation $v_1, v_2 \ldots, v_n$ of the vertices, which represents $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \rightarrow v_1$

Example



- For simplicity, examples will generally be drawn undirected
- Imagine each edge (i, j) is really two parallel edges with same cost
- Optimal tour cost:

  10 + 25 + 30 + 15 = 80

Attempt: Solving TSP with CP?

- Define 0/1 variables x_{ij} indicating if edge (i, j) is in the TSP tour
- Each vertex is visited exactly once:

$$\sum_{j \neq i} x_{ij} = \sum_{j \neq i} x_{ji} = 1, \qquad \forall 1 \leq i \leq n$$

- Want to minimize total cost:

$$C = \sum_{(i,j)} w(i, j) \cdot x_{ij}$$

An issue

- This CP formulation allows “subtours” rather than forcing one contiguous tour!

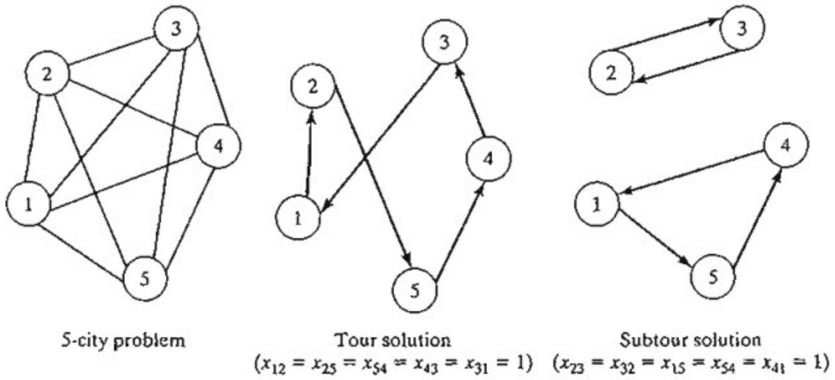


FIGURE 9.11

A 5-city TSP example with a tour and subtour solutions of the associated assignment model

Disallowing subtours

- For each possible subtour of vertices S, make sure that we take less than $|S|$ edges between them
- As a constraint:

$$\sum_{i \neq j \in S} x_{ij} < |S|, \qquad \forall S \subset V, |S| > 1$$

- Problem: there are exponentially many subtours!
  - Ways to fix this or add constraints lazily…
  - But in general CP is not state-of-the-art for TSP

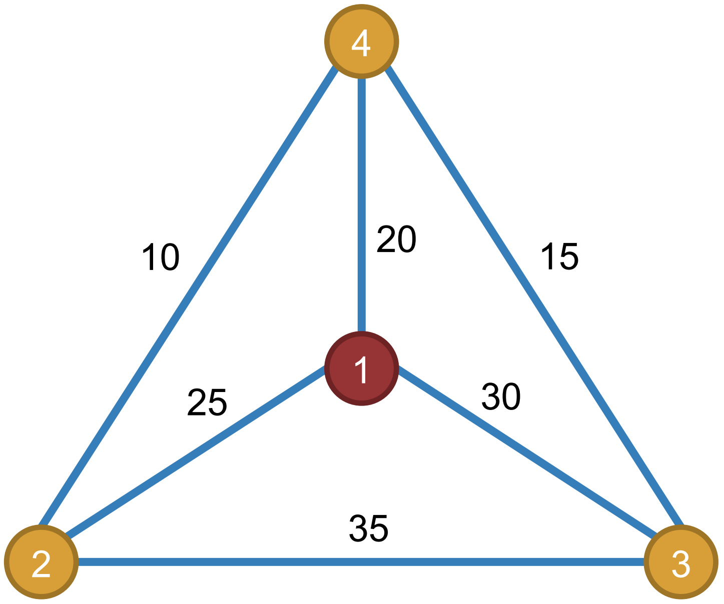
Traveling Salesman Problem

- Observation: TSP is an approximation-friendly problem
    - In practice, "good enough" usually is good enough!
- **Goal:** design efficient heuristics that give an empirically cheap tour (possibly not quite cheap*est*)
- Today: **constructive heuristics**
    - Start from nothing and iteratively build up partial solution

Nearest-Neighbor (NN)

- Start at any vertex u. Pick nearest unseen out-neighbor v of u and add it to end of tour, then repeat starting from v. Continue until all vertices added.
- Pros:
  - Simple, intuitive, and relatively efficient
  - Empirically OK, esp. on Euclidean TSP
- Cons:
  - Greedy: can easily miss shortcut paths

Nearest-Neighbor (NN)



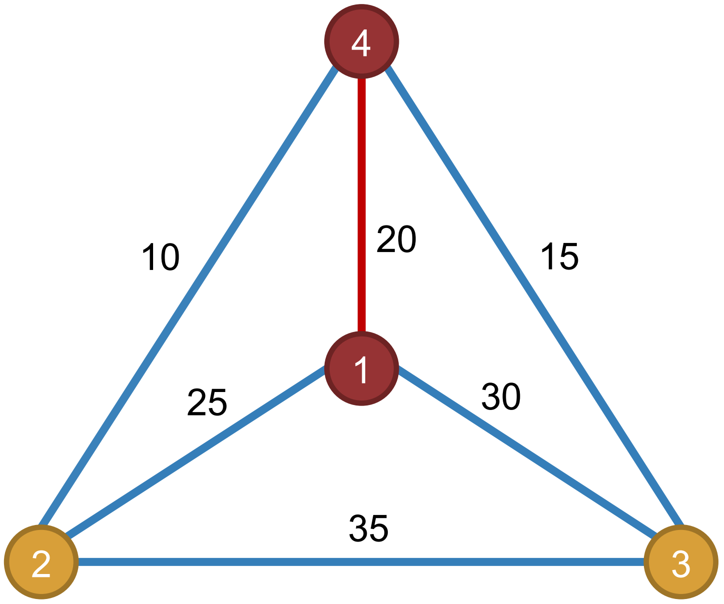
- Current tour:

  1

- Current cost:

  0

Nearest-Neighbor (NN)



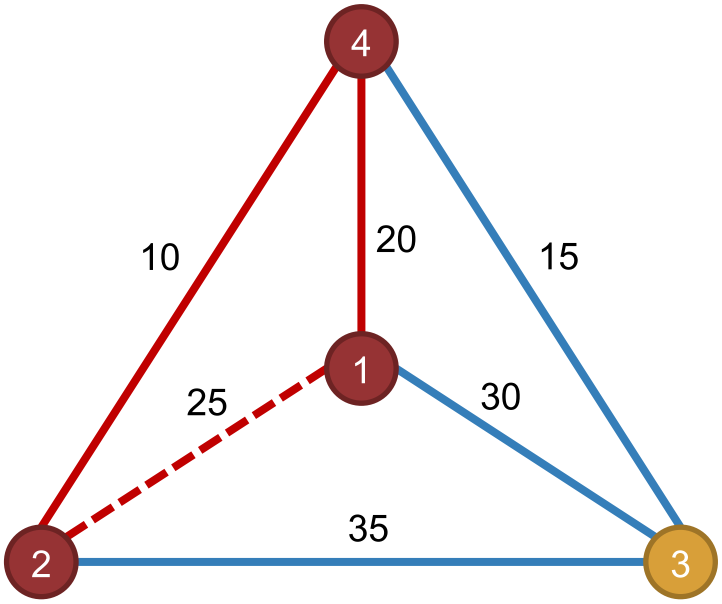
- Current tour:

  1, 4

- Current cost:

  20 + 20 = 40

Nearest-Neighbor (NN)



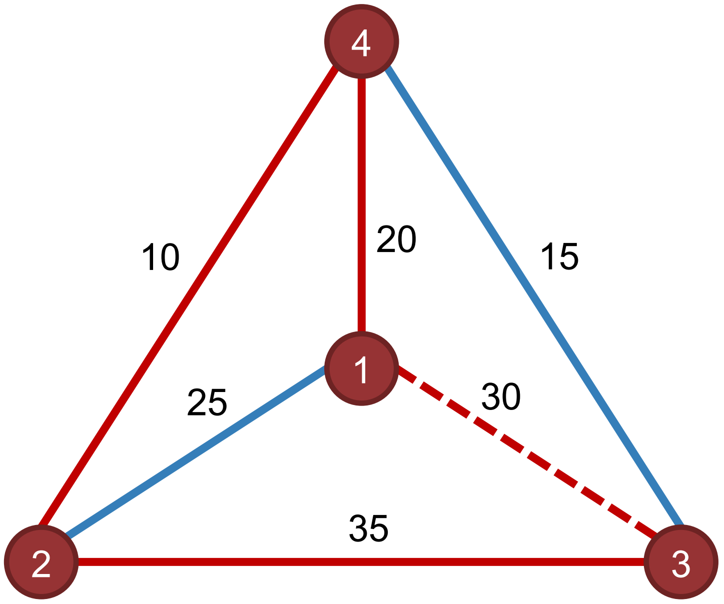
- Current tour:

  1, 4, 2

- Current cost:

  20 + 10 + 25 = 55

Nearest-Neighbor (NN)



- Current tour:

  1, 4, 2, 3

- Current cost:

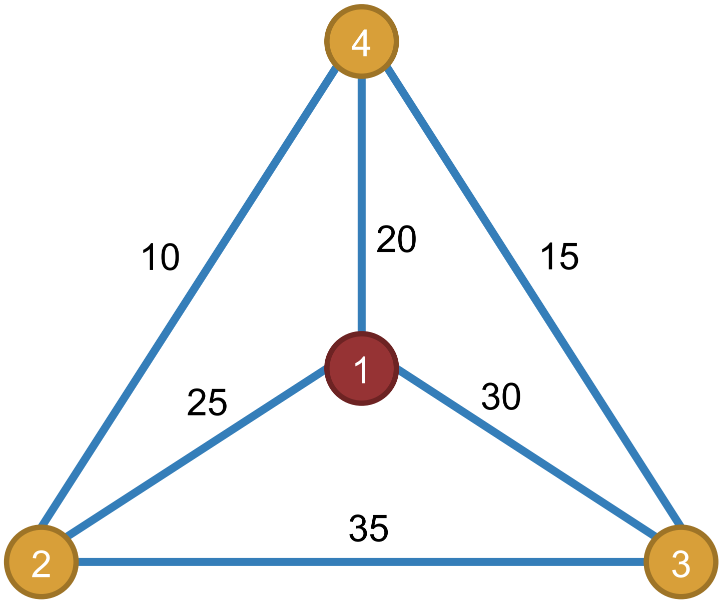
  20 + 10 + 35 + 30 = 95

Nearest-Insertion (NI)

- Start the tour T at any vertex
- Pick the nearest unseen out-neighbor v of **any** vertex in the tour
- Insert it into the tour $T = t_1, \dots, t_k$ so that the total tour distance is minimized
  - i.e., find i s.t. $w(t_i, v) + w(v, t_{i+1}) - w(t_i, t_{i+1})$ is minimized
- Repeat until all vertices added to tour

- **Intuition:** still greedy, but not as greedy as NN – allow the partial tour to be modified

Nearest-Insertion (NI)



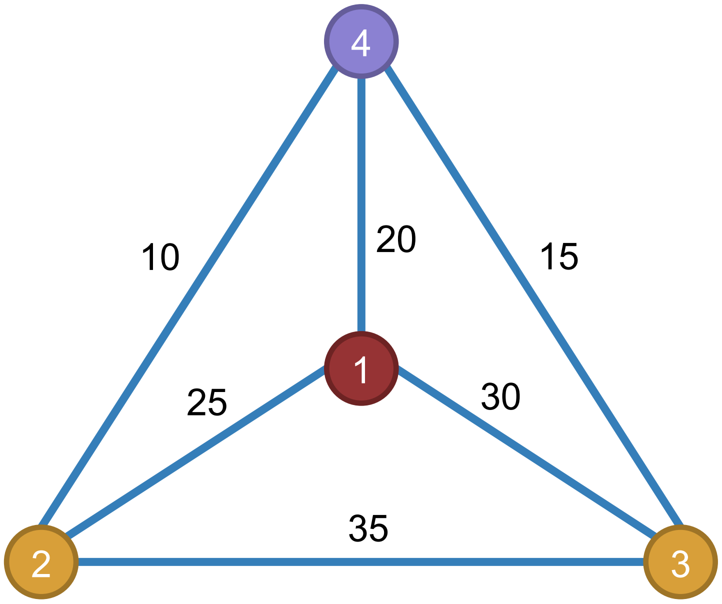
- Current tour:

  1

- Current cost:

  0

Nearest-Insertion (NI)



- Current tour:

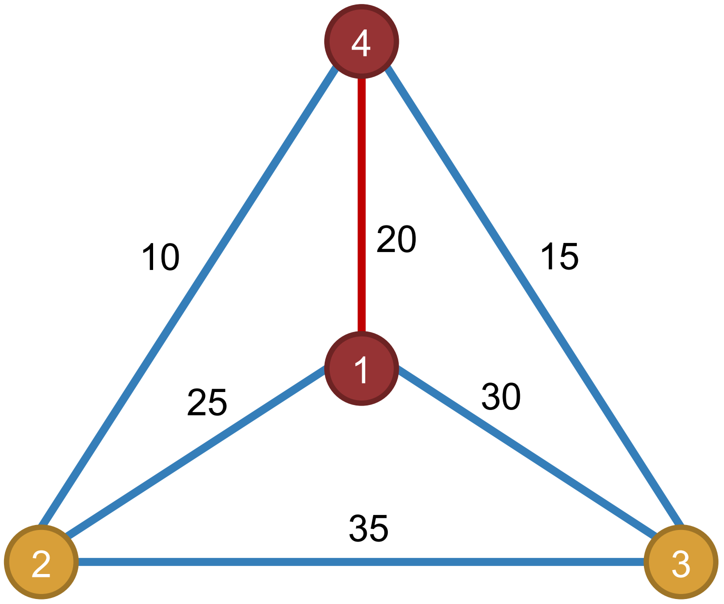
    1

- Current cost:

    0

- Next vertex: 4
    - Only one place to insert (up to rotation)

Nearest-Insertion (NI)



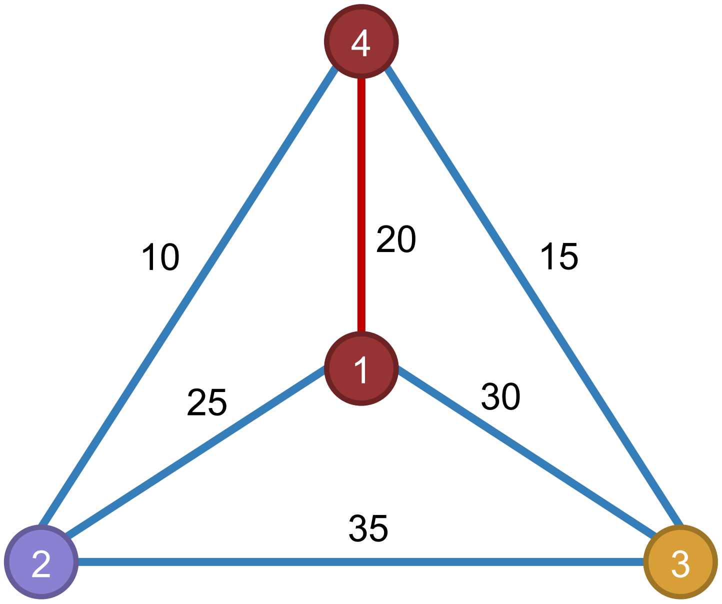
- Current tour:

  1, 4

- Current cost:

  20 + 20 = 40

Nearest-Insertion (NI)



- Current tour:

  1, 4

- Current cost:

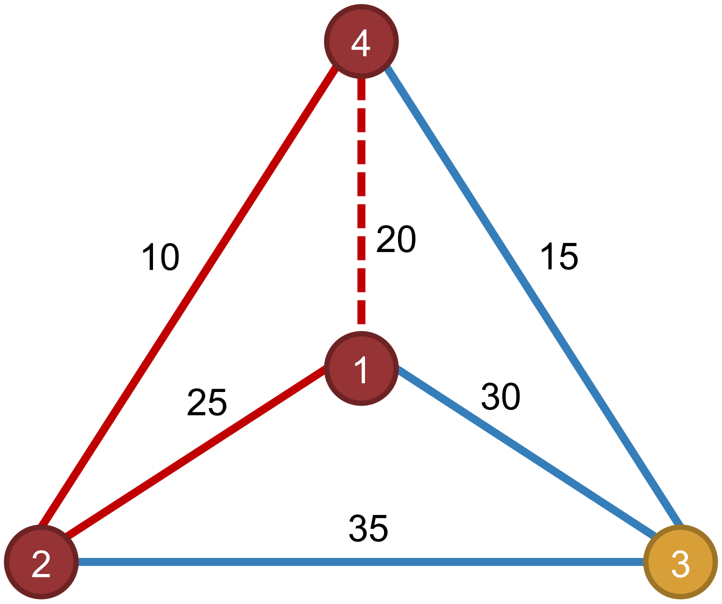
  20 + 20 = 40

- Next vertex: 2

After 1: $w(1, 2) + w(2, 4) - w(1,4) = 25 + 10 - 20 = 15$

After 4: $w(4, 2) + w(2, 1) - w(1,4) = 10 + 25 - 20 = 15$

Nearest-Insertion (NI)



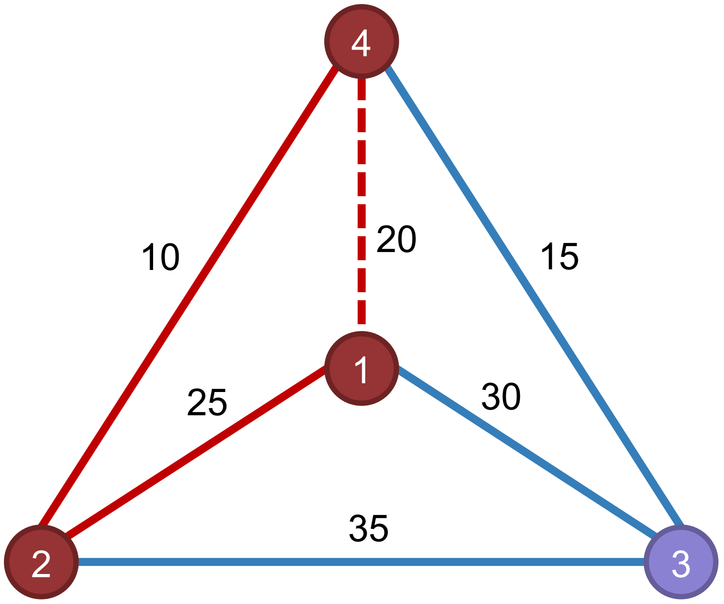
- Current tour:

  1, 2, 4

- Current cost:

  25 + 10 + 20 = 55

Nearest-Insertion (NI)



- Current tour:

  1, 2, 4

- Current cost:

  25 + 10 + 20 = 55

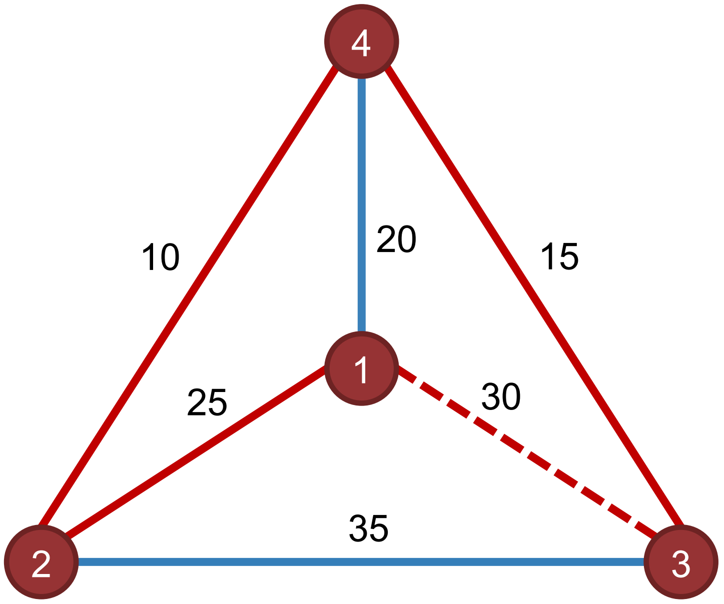
- Next vertex: 3

After 1: $w(1, 3) + w(3, 2) - w(1, 2) = 30 + 35 - 25 = 40$

After 2: $w(2, 3) + w(3, 4) - w(2, 4) = 35 + 15 - 10 = 40$

After 4: $w(4, 3) + w(3, 1) - w(4, 1) = 15 + 30 - 20 = 35$

Nearest-Insertion (NI)



- Current tour:

  1, 2, 4, 3

- Current cost:

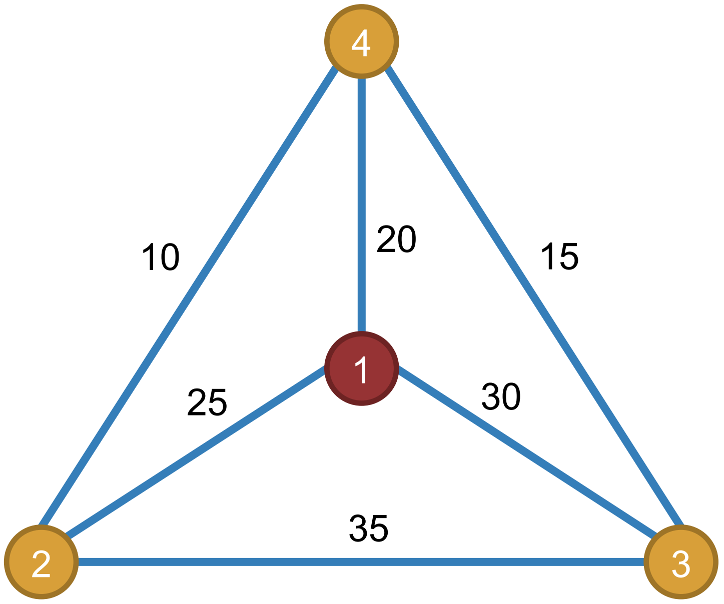
  25 + 10 + 15 + 30 = 80

Farthest-Insertion (FI)

- Start the tour T at any vertex
- Pick the ~~nearest~~ farthest unseen out-neighbor v of **any** vertex in the tour
- Insert it into the tour $T = t_1, \ldots, t_k$ so that the total tour distance is minimized
  - i.e., find i s.t. $w(t_i, v) + w(v, t_{i+1}) - w(t_i, t_{i+1})$ is minimized
- Repeat until all vertices added to tour

- **Intuition:** start with the general outline of the tour and then fill in the details later

Farthest-Insertion (FI)



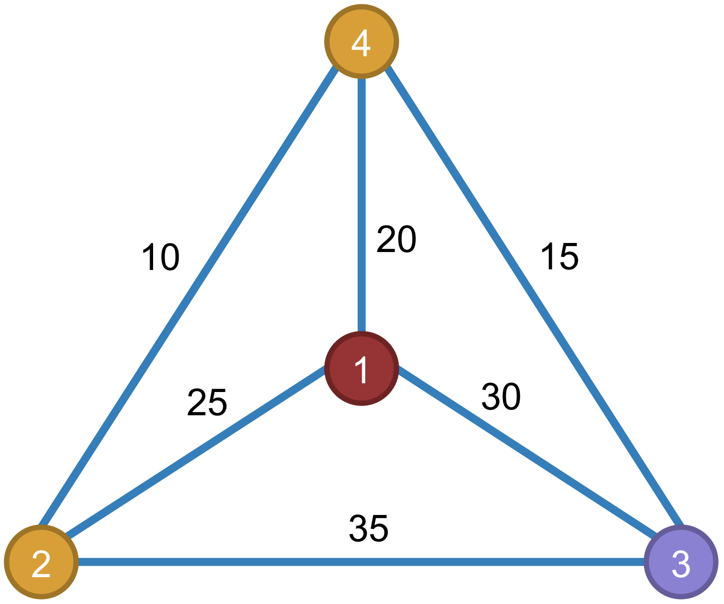
- Current tour:

  1

- Current cost:

  0

Farthest-Insertion (FI)



- Current tour:

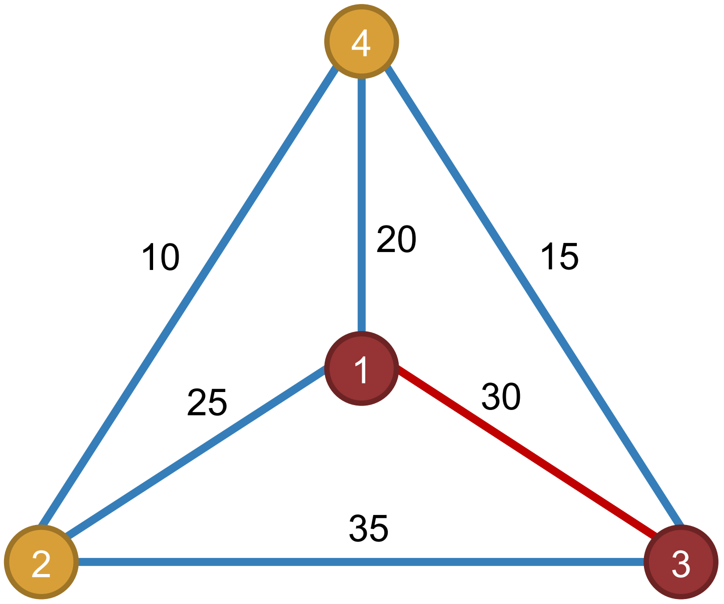
  1

- Current cost:

  0

- Next vertex: 3
  - Only one place to insert (up to rotation)

Farthest-Insertion (FI)



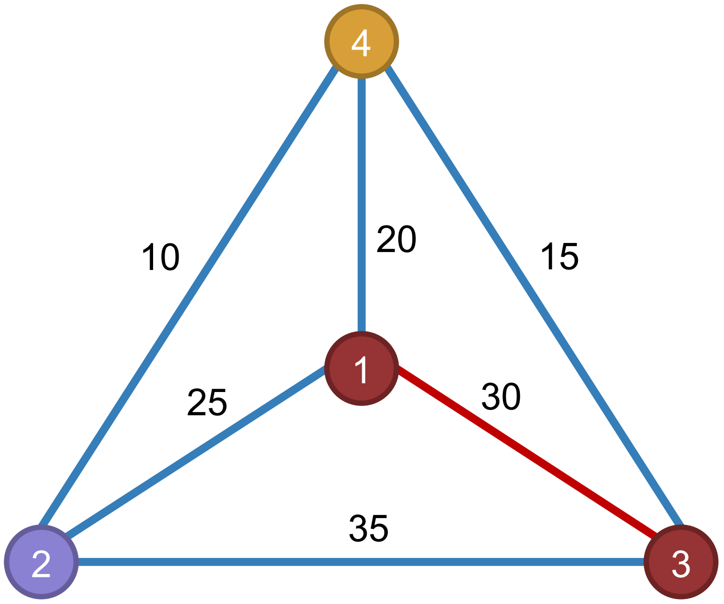
- Current tour:

  1, 3

- Current cost:

  30 + 30 = 60

Farthest-Insertion (FI)



- Current tour:

  1, 3

- Current cost:

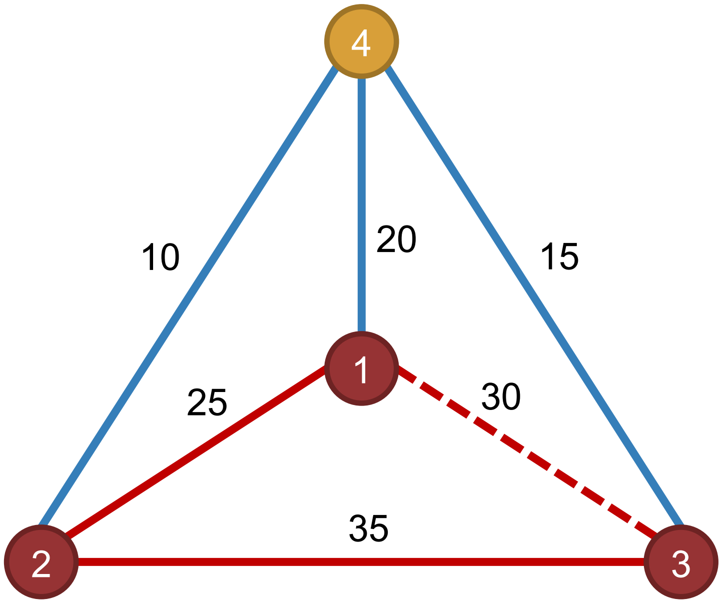
  30 + 30 = 60

- Next vertex: 2

After 1: $w(1, 2) + w(2, 3) - w(1, 3) = 25 + 35 - 30 = 30$

After 3: $w(3, 2) + w(2, 1) - w(1, 3) = 35 + 25 - 30 = 30$

Farthest-Insertion (FI)



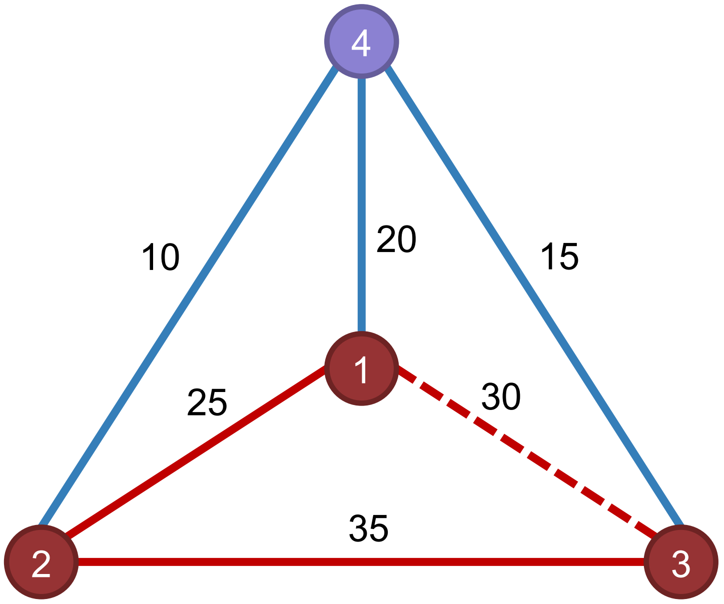
- Current tour:

  1, 2, 3

- Current cost:

  25 + 35 + 30 = 90

Farthest-Insertion (FI)



- Current tour:

  1, 2, 3

- Current cost:

  25 + 35 + 30 = 90

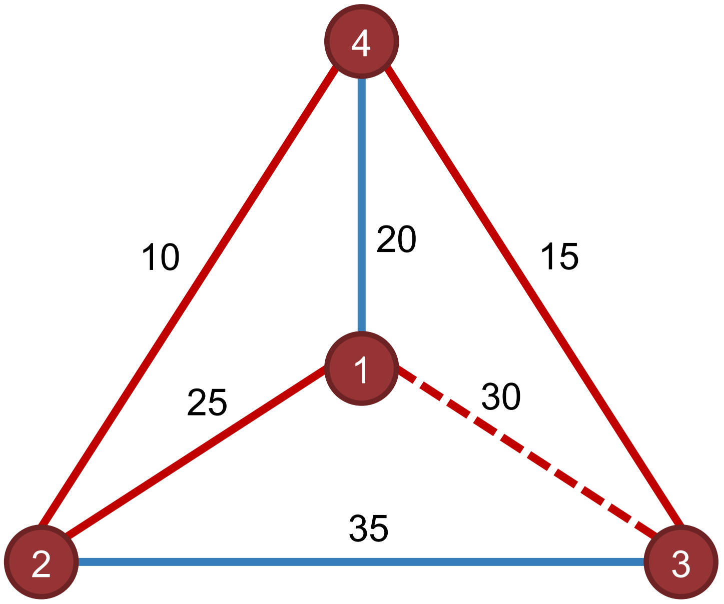
- Next vertex: 4

After 1: $w(1, 4) + w(4, 2) - w(1, 2) = 20 + 10 - 25 = 5$

After 2: $w(2, 4) + w(4, 2) - w(2, 3) = 10 + 15 - 35 = -10$

After 3: $w(3, 4) + w(4, 1) - w(3, 1) = 15 + 20 - 30 = 5$

Farthest-Insertion (FI)



- Current tour:

  1, 2, 4, 3

- Current cost:

  25 + 10 + 15 + 30 = 80

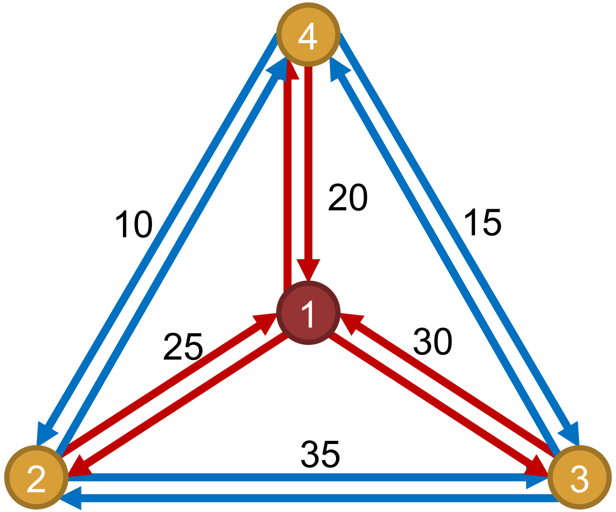
Insertion Heuristics

- Aims to be less naively greedy than NN
  - Unlike NN, can modify partial tour
- Somewhat more expensive than NN heuristic
- FI works pretty well in practice...
- ...but NI not so much.

Savings Heuristic

- Pick any vertex x to be the "central vertex"
- Start with $n-1$ subtours: $x \to v \to x$ for all $v \in V - x$
- For each edge (i, j), where $i, j \in V - x$, compute its **savings** $s(i, j)$
  - $s(i, j) = w(i, x) + w(x, j) - w(i, j)$
- Sort edges in decreasing order of savings
- Repeat until only one tour remains:
- Let (i, j) be the next edge in sorted order
- If edges (i, x) and (x, j) are in our subtours, and i, j are not already in the same tour: replace (i, x) and (x, j) by (i, j)

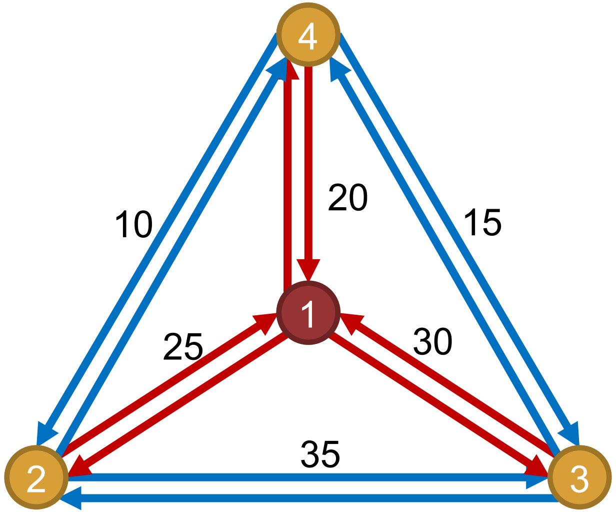
Savings Heuristic



- Current cost:

25 + 25 + 30 + 30 + 20 + 20 = 150

Savings Heuristic

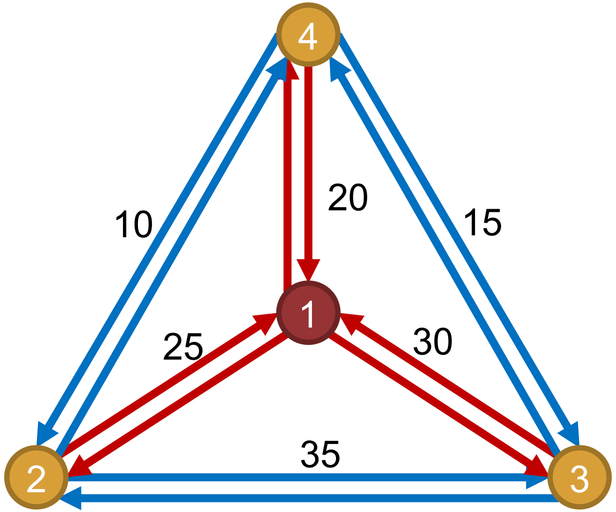


- Current cost:

25 + 25 + 30 + 30 + 20 + 20 = 150

(i,j)	Savings $s(i,j)$
$(2,3)$	$w(2,1)+w(1,3)-w(2,3)$ $=25+30-35=20$
$(3,2)$	$w(3,1)+w(1,2)-w(3,2)$ $=30+25-35=20$
$(2,4)$	$w(2,1)+w(1,4)-w(2,4)$ $=25+20-10=35$
$(4,2)$	$w(4,1)+w(1,2)-w(4,2)$ $=20+25-10=35$
$(3,4)$	$w(3,1)+w(1,4)-w(3,4)$ $=30+20-15=35$
$(4,3)$	$w(4,1)+w(1,3)-w(4,3)$ $=20+30-15=35$

Savings Heuristic

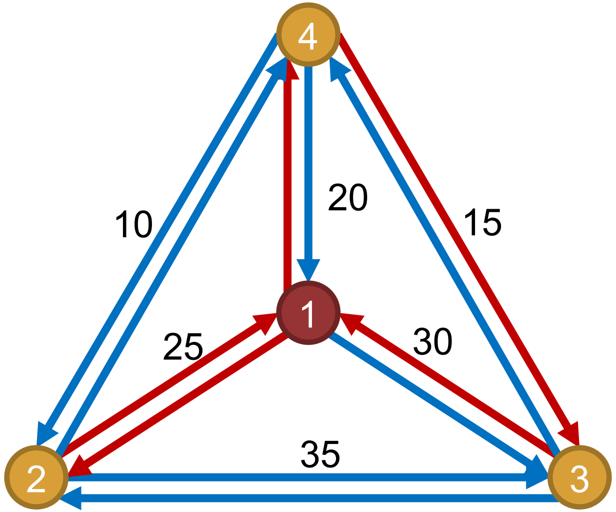


- Current cost:

25 + 25 + 30 + 30 + 20 + 20 = 150

(i, j)	Savings $s(i, j)$
$(2, 3)$	$w(2,1) + w(1,3) - w(2,3)$ $= 25 + 30 - 35 = 20$
$(3, 2)$	$w(3,1) + w(1,2) - w(3,2)$ $= 30 + 25 - 35 = 20$
$(2, 4)$	$w(2,1) + w(1,4) - w(2,4)$ $= 25 + 20 - 10 = 35$
$(4, 2)$	$w(4,1) + w(1,2) - w(4,2)$ $= 20 + 25 - 10 = 35$
$(3, 4)$	$w(3,1) + w(1,4) - w(3,4)$ $= 30 + 20 - 15 = 35$
$(4, 3)$	$w(4,1) + w(1,3) - w(4,3)$ $= 20 + 30 - 15 = 35$

Savings Heuristic

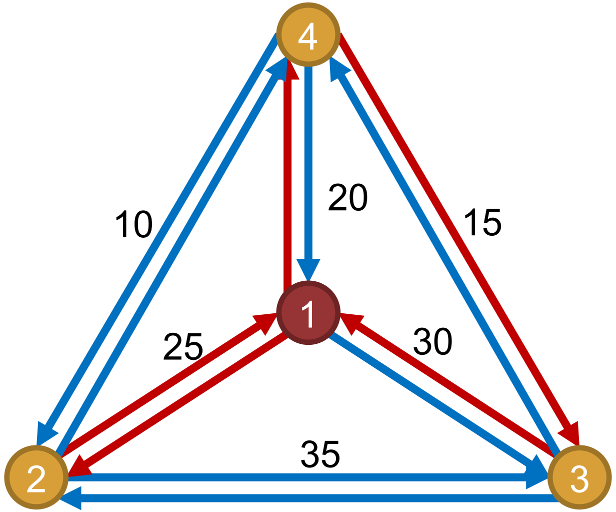


- Current cost:

25 + 25 + 20 + 15 + 30 = 115

(i,j)	**Savings** $s(i,j)$
$(2,3)$	$w(2,1)+w(1,3)-w(2,3)$ $=25+30-35=20$
$(3,2)$	$w(3,1)+w(1,2)-w(3,2)$ $=30+25-35=20$
$(2,4)$	$w(2,1)+w(1,4)-w(2,4)$ $=25+20-10=35$
$(4,2)$	$w(4,1)+w(1,2)-w(4,2)$ $=20+25-10=35$
$(3,4)$	$w(3,1)+w(1,4)-w(3,4)$ $=30+20-15=35$
$(4,3)$	$w(4,1)+w(1,3)-w(4,3)$ $=20+30-15=35$

Savings Heuristic



- Current cost:

25 + 25 + 20 + 15 + 30 = 115

(i,j)	Savings $s(i,j)$
$(2,3)$	$w(2,1)+w(1,3)-w(2,3)$ $=25+30-35=20$
$(3,2)$	$w(3,1)+w(1,2)-w(3,2)$ $=30+25-35=20$
$(2,4)$	$w(2,1)+w(1,4)-w(2,4)$ $=25+20-10=35$
$(4,2)$	$w(4,1)+w(1,2)-w(4,2)$ $=20+25-10=35$
$(3,4)$	$w(3,1)+w(1,4)-w(3,4)$ $=30+20-15=35$
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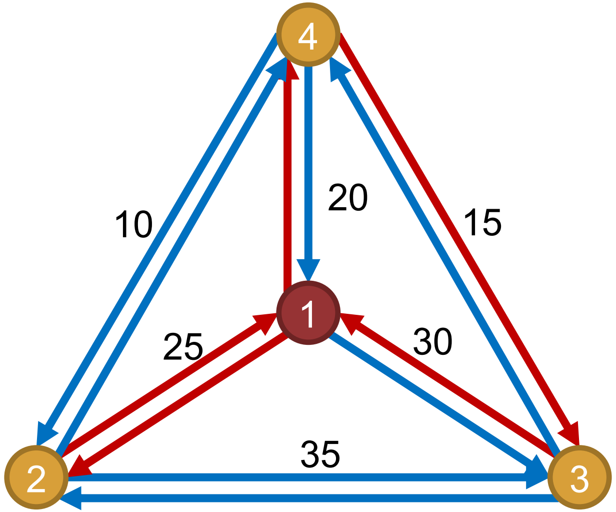
Savings Heuristic

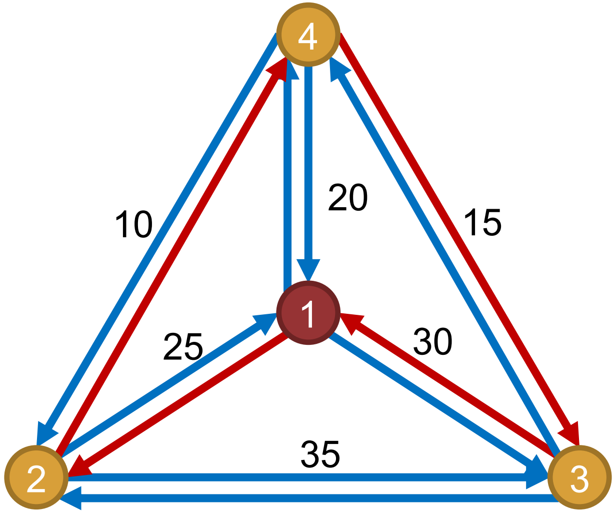


- Current cost:

25 + 25 + 20 + 15 + 30 = 115

(i, j)	Savings $s(i, j)$
$(2, 3)$	$w(2,1) + w(1,3) - w(2,3)$ $= 25 + 30 - 35 = 20$
$(3, 2)$	$w(3,1) + w(1,2) - w(3,2)$ $= 30 + 25 - 35 = 20$
$(2, 4)$	$w(2,1) + w(1,4) - w(2,4)$ $= 25 + 20 - 10 = 35$
$(4, 2)$	$w(4,1) + w(1,2) - w(4,2)$ $= 20 + 25 - 10 = 35$
$(3, 4)$	$w(3,1) + w(1,4) - w(3,4)$ $= 30 + 20 - 15 = 35$
$(4, 3)$	$w(4,1) + w(1,3) - w(4,3)$ $= 20 + 30 - 15 = 35$

Savings Heuristic



- Current cost:

25 + 10 + 15 + 30 = 80

(i, j)	Savings $s(i, j)$
$(2,3)$	$w(2,1) + w(1,3) - w(2,3)$ $= 25 + 30 - 35 = 20$
$(3,2)$	$w(3,1) + w(1,2) - w(3,2)$ $= 30 + 25 - 35 = 20$
$(2,4)$	$w(2,1) + w(1,4) - w(2,4)$ $= 25 + 20 - 10 = 35$
$(4,2)$	$w(4,1) + w(1,2) - w(4,2)$ $= 20 + 25 - 10 = 35$
$(3,4)$	$w(3,1) + w(1,4) - w(3,4)$ $= 30 + 20 - 15 = 35$
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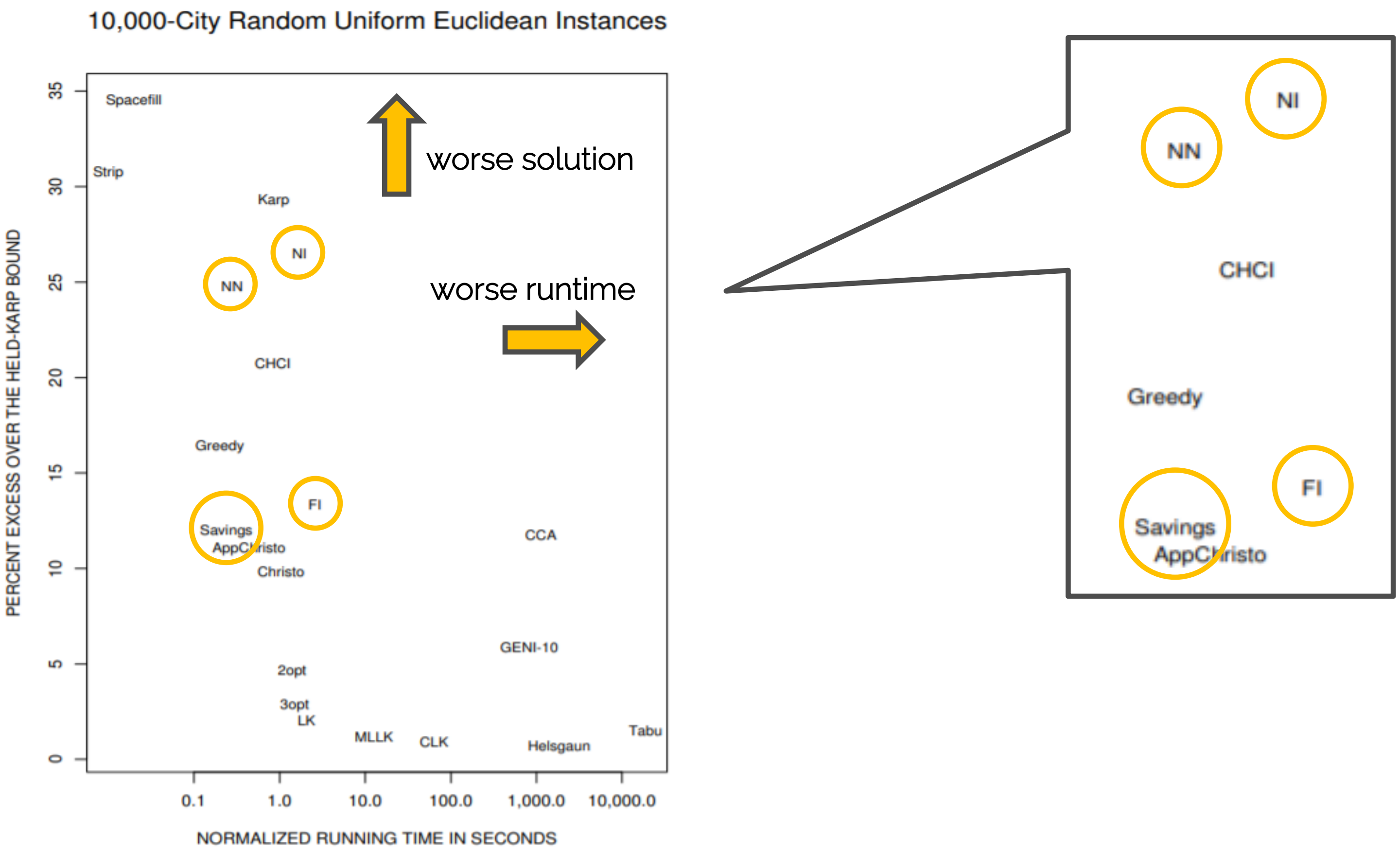
10,000-City Random Uniform Euclidean Instances
PERCENT EXCESS OVER THE HELD-KARP BOUND
NORMALIZED RUNNING TIME IN SECONDS
0
5
10
15
20
25
30
35
0.1
1.0
10.0
100.0
1,000.0
10,000.0
Spacefill
Strip
Karp
NI
NN
CHCI
Greedy
FI
Savings
AppChristo
Christo
CCA
GENI-10
2opt
3opt
LK
MLLK
CLK
Helsgaun
Tabu
worse solution
worse runtime

Vehicle Routing Problem

- Actually, the Savings heuristic was created to solve a generalization of the TSP:
- The **Vehicle Routing Problem (VRP)** also takes place in a weighted, complete graph
- Instead of one salesman, we have a fleet of vehicles which are all parked at a central vertex (the **depot**)
    - May or may not be a limit on the number of vehicles
- **Goal:** find routes starting and ending at the depot for each vehicle with minimum total weight so that each vertex is visited once by some vehicle

Constrained VRP

- In real life: why use a fleet of vehicles when you could have one vehicle that travels all the routes?
- There may be additional constraints for vehicles, e.g.:
  - Maximum distance a vehicle can travel
  - Carrying capacity of a vehicle, where each node has some volume to be delivered

Savings Heuristic for VRP

- Let x denote the depot
- Start with $n - 1$ subtours: $x \to v \to x$ for all $v \in V - x$
- For each edge (i, j), where $i, j \in V - x$, compute its **savings** $s(i, j)$
  - $s(i, j) = w(i, x) + w(x, j) - w(i, j)$
- Sort edges in decreasing order of savings
- Repeat until only one tour remains or we reach negative savings:
- Let (i, j) be the next edge in sorted order
- If edges (i, x) and (x, j) are in our subtours, and i, j are not already in the same tour: replace (i, x) and (x, j) by (i, j)…
  - …unless it would violate our constraints

Solving TSP with OR-Tools

- OR-Tools comes with a **routing solver** that can solve the TSP and VRP with much more complex constraints!
  - Pickups and drop-offs, time windows, penalties...
- The guide is pretty good: https://developers.google.com/optimization/routing
- Comes with many heuristics including NN, Savings, etc...
  - By default, solver automatically chooses a heuristic to use based on the problem at hand
- Note: the routing solver is optimized for getting a "good enough" solution to constrained problems, not exact solving huge TSPs

Scaling and Shifting

- Warning: the OR-Tools routing solver may not work correctly with fractional/negative edge weights
  - Even worse, it might not throw an error!
- Can fix negative weights by **shifting**:
  - Add large constant K to all weights to make them positive
  - Preserves TSP structure since all tours increase by $K \cdot n$
  - May not necessarily preserve VRP structure ¯_(ツ)_/¯
- Can fix fractional weights by **scaling**:
  - Multiply all weights by a large constant M to make them integers (or minimize rounding error)
  - If no rounding, preserves TSP and VRP structure

The OR-Tools TSP Solver **doesn't** always produce an optimal solution.

How well does it do **in practice**?

Let's test it on instances from the National TSP Collection, a set of real-world instances ranging in size from 29 to 71,000+ nodes.

Benchmarking the TSP Solver

Country	# Cities	Output Cost	Optimal Cost	Percent Error	*Runtime (s)
W. Sahara	29	27749	27603	0.53%	0.0320
Djibouti	38	7078	6656	6.3%	0.0657
Qatar	194	10064	9352	7.6%	2.61
Uruguay	734	83476	79114	5.5%	37.9
Zimbabwe	929	101100	95345	6.0%	91.4
Oman	1979	92250	86891	6.2%	668

*Running on my Dell XPS laptop with 16GB of RAM, in a Jupyter notebook.