CIS 1600

Recitation Guide - Week 8

Topics Covered: Expectation

Problem 1: Elisa, as a busy college student, hasn't done laundry in weeks. In particular, she realizes that she has no more socks to wear, so she goes to the laundry room and throws in her 2 distinct pairs of socks. However, the machine is broken, and so it only returns 2 of her socks at random! Note that the two socks in a pair are also distinguishable.

(a) What is the expected number of pairs that she can wear now?

(b) What if she throws in n pairs and only gets k ($k > 1$) socks back?

Solution:

(a) We will denote the number of pairs she can wear using random variable S. Since only 2 socks are returned, S can only take on values 0 and 1. We define the sample space $\Omega = \{\{s_1, s_2\} \mid s_i \text{ is a sock}\}$. Notice that the sample space is uniform. Then we have

$$Pr[S = 0] = \frac{\binom{2}{1} \cdot \binom{2}{1}}{\binom{4}{2}} = \frac{2}{3}$$

$$Pr[S = 1] = \frac{2 \cdot \binom{2}{2}}{\binom{4}{2}} = \frac{1}{3}$$

Hence,

$$E[S] = 0 \cdot Pr[S = 0] + 1 \cdot Pr[S = 1] = \frac{1}{3}$$

(b) If she throws in n pairs, let $I_1, I_2, \ldots, I_n$ be indicator random variables where $I_i = 1$ if the pair i is returned to Elisa (and $I_i = 0$ if the pair is not). Now, we want to find $E[S]$. Notice that

$$\begin{aligned} E[S] &= E\left[\sum_{i=1}^{n} I_i\right] \\ &= \sum_{i=1}^{n} E[I_i] \qquad \text{by Linearity of Expectation} \end{aligned}$$

For an arbitrary I_i, notice that $E[I_i] = \Pr[I_i = 1] = \Pr[E_i]$, where E_i is the event that the i^{th} pair was returned. We define the sample space $\Omega = \{\{s_1, s_2, \ldots, s_k\} \mid s_i \text{ is a sock}\}$. Notice that the sample space is uniform, since each subset of k socks are equally likely to be returned.

Since the sample space is uniform, we have that $\Pr[E_i] = \frac{|E_i|}{\binom{2n}{k}}$. We also know that $|E_i| = \binom{2n-2}{k-2}$, the number of ways of returning k socks that includes pair i. Thus, we have

$$
\begin{aligned}
E[S] &= \sum_{i=1}^{n} E[I_i] \\
&= n \cdot \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}}
\end{aligned}
$$

Problem 2: Winnie starts the month with 500 dollars in her bank account. For each day i in the month of March ($i = 1, 2, \ldots, 31$), Winnie flips a fair coin. If the coin flip results in heads, Winnie takes i dollars out of her bank account. If the coin flip results in tails, Winnie deposits 10 dollars into her bank account. What is the expected amount in the account at the end of March? You do not need to define the probability space.

Solution:

Let the random variable Y denote the amount in Winnie's bank account at the end of the month. Let the random variable X denote the net change in Winnie's bank account at the end of March. Observe that $Y = X + 500$. Thus, we can see that what we are looking for is $E[Y] = E[500 + X]$. By Linearity of Expectation, we see that $E[500 + X] = E[500] + E[X]$. The expected value of a constant is the constant itself, so we only need to find $E[X]$.

Let X_i be the net change in Winnie's bank account on the ith day. We observe that

$$X = \sum_{i=1}^{31} X_i$$

Thus, we can calculate the expectation of X as follows:

$$\begin{aligned}
E[X] &= E[\sum_{i=1}^{31} X_i] \\
&= \sum_{i=1}^{31} E[X_i] && \text{(by Linearity of Expectation)} \\
&= \sum_{i=1}^{31} (\frac{1}{2} \cdot (-i) + \frac{1}{2} \cdot 10) && \text{(definition of expectation)} \\
&= \sum_{i=1}^{31} (\frac{-i}{2} + 5) \\
&= 155 + \sum_{i=1}^{31} \frac{-i}{2} \\
&= 155 - \frac{31 \cdot 32}{4} && \text{(applying } \textstyle\sum_{i=1}^{n} = \frac{n(n+1)}{2}\text{)} \\
&= 155 - 248 \\
&= -93
\end{aligned}$$

Therefore, the expected value in Winnie's bank account at the end of the month is $E[Y] = E[500] + E[X] = 500 - 93 = 407$ dollars.