CIS 1600

# Recitation Guide - Week 8

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**Topics Covered:** Graphs, Trees, Independence

**Problem 1:**
We have three wooden buckets, $A, B, C$ and we throw $n \geq 3$ metal keys in them. The key throws are mutually independent and each key is equally likely to land in each of the three buckets.

(a) Let $A$ be the event that after all keys are thrown, bucket $A$ has at least one key in it and similarly associate an event $B$ with $B$. Are $A$ and $B$ independent? Justify your answer.

(b) Compute the probability that after all keys are thrown, each of the three buckets has at least one key in it. Justify your answer.

**Solution:**

Define $\Omega = \{(x_1, x_2, ..., x_n) \mid x_i \in \{A, B, C\}\}$. That is, the sample space is the set of all $n$-tuples where the $i$th element in the tuple represents the bucket that key $i$ landed in.

(a) For $i = 1, \ldots, n$ let $A_i$ be the event that key $i$ is thrown in bucket $A$. We have $\Pr[A_i] = \dfrac{1}{3}$. Clearly $A = A_1 \cup \cdots \cup A_n$ and since the events $A_1, \ldots, A_n$ are mutually independent we can compute:

$$\Pr[A] = \Pr[A_1 \cup \cdots \cup A_n] \;=\; 1 \;-\; \prod_{i=1}^{n}(1 - \Pr[A_i]) \;=\; 1 - \left(1 - \frac{1}{3}\right)^n \;=\; 1 - \left(\frac{2}{3}\right)^n$$

Similarly, $\Pr[B] = 1 - \left(\dfrac{2}{3}\right)^n$. To check independence we also need $\Pr[A \cap B]$.

Upon reflection, we notice that there is one aspect of the problem that we have not used yet: the keys get thrown *only* in $A, B$ and $C$. Thus, $\bar{A} \cap \bar{B}$, which means that both $A$ and $B$ are empty after all keys are thrown, is the same as the event "all keys get thrown in $C$" and therefore, by mutual independence, has probability $\left(\dfrac{1}{3}\right)^n$, as each key has a $\dfrac{1}{3}$ probability of being thrown into $C$. Now we can compute, using properties of probability and De Morgan's Laws:

$$\begin{aligned}
\Pr[A \cap B] &= \Pr[A] + \Pr[B] - \Pr[A \cup B] \\
&= \Pr[A] + \Pr[B] - \left(1 - \Pr[\overline{A \cup B}]\right) \\
&= \Pr[A] + \Pr[B] - \left(1 - \Pr[\bar{A} \cap \bar{B}]\right) \\
&= 1 - \left(\frac{2}{3}\right)^n + 1 - \left(\frac{2}{3}\right)^n - \left(1 - \left(\frac{1}{3}\right)^n\right) \\
&= 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n
\end{aligned}$$

We also know that:

$$\Pr[A] \cdot \Pr[B] = \left(1 - \left(\frac{2}{3}\right)^n\right)\left(1 - \left(\frac{2}{3}\right)^n\right) = 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{4}{9}\right)^n$$

Now we check for independence,

$$\Pr[A \cap B] \stackrel{?}{=} \Pr[A] \cdot \Pr[B]$$
$$1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n \neq 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{4}{9}\right)^n$$

because $\frac{1}{3} \neq \frac{4}{9}$, and so $A$ and $B$ are not independent.

(b) We continue with the notation introduced in part (a) and we also define $C$ to be the event "$C$ is not empty after all keys are thrown." This part asks for $\Pr[A \cap B \cap C]$. We are tempted to multiply probabilities but we do not know if $A, B, C$ are mutually independent. In fact, in part (a) we saw that $A \not\perp B$. Although it is still possible that $\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$ there is no reason to hope for this here (and in fact we shall see that it does not hold).

Instead, we will use the Principle of Inclusion-Exclusion for three events:

$$\Pr[A \cup B \cup C] = \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[C \cap A] + \Pr[A \cap B \cap C]$$

Since we have at least one key, at least one of the buckets ends up non-empty. Hence $A \cup B \cup C = \Omega$, meaning $A \cup B \cup C$ consists of all the outcomes and has probability 1. From part (a) we have:

$$\Pr[A] = \Pr[B] = \Pr[C] = 1 - \left(\frac{2}{3}\right)^n$$
$$\Pr[A \cap B] = \Pr[B \cap C] = \Pr[C \cap A] = 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n$$

We plug in and obtain

$$\begin{aligned}
\Pr[A \cap B \cap C] &= \Pr[A \cup B \cup C] - \Pr[A] - \Pr[B] - \Pr[C] + \Pr[A \cap B] + \Pr[B \cap C] + \Pr[C \cap A] \\
&= 1 - 3\left(1 - \left(\frac{2}{3}\right)^n\right) + 3\left(1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n\right) \\
&= 1 - 3\left(\frac{2}{3}\right)^n + 3\left(\frac{1}{3}\right)^n
\end{aligned}$$