CIS 1600

Recitation Guide - Week 7

Topics Covered: Graphs, Trees, Independence

Problem 1: Let T be a tree where the maximum degree is Δ. Prove that T has at least Δ leaves by contradiction.

Solution:

Assume that $\Delta \geq 2$, since the cases of $\Delta = 0$ and $\Delta = 1$ are clearly true. Suppose for the sake of contradiction that there are at most $\psi < \Delta$ leaves. Let $v \in V$ have degree Δ. Consider $S = \{u \in V \mid \{u, v\} \in E\}$. Note that S is the set of v's neighbors, and $|S| = \Delta$.

For all $u_i \in S$, there exists at least one path that starts with $\{v, u_i\}$ that ends with a leaf. We pick any such leaf for each edge $\{v, u_i\}$ and call the leaf l_i. Note there is a unique l_i corresponding to each u_i, as trees are acyclic, so we have Δ l_i's in total. Hence, by the Pigeonhole Principle, where the pigeons are the terminating leaves l_i of each path and the holes are the ψ leaves available, we know that $\lceil \frac{\Delta}{\psi} \rceil \geq \lceil \frac{\Delta}{\Delta - 1} \rceil = \lceil 1 + \frac{1}{\Delta - 1} \rceil$ (since $\Delta \geq 2$) $= 2$ paths share the same terminating leaf, say ℓ_ω.

This is a contradiction, since the path between ℓ_ω and v are unique in a tree.

For each $u_i \in S$, let p_i be a maximal path starting from $v - u_i$. Note that there must be Δ such paths. We know from the lemma proven above that all such p_i must terminate in a leaf ℓ_i.

Problem 2:
Prove that G or the complement of G is connected. Note that the complement of a graph $G = (V, E)$ is $G^c = (V, E')$ and $\forall u, v \in V$, $\{u, v\} \in E' \iff \{u, v\} \notin E$.

Solution:

If G is connected we are done.

If G is not connected then G is composed of multiple connected components. We want to prove that given two arbitrary vertices in G there must be a path between them in G^c. Let these two arbitrary vertices be u and v.

Case 1: u and v do not share an edge in G

This means they must share an edge in G^c and thus there is a path from u to v in G^c.

Case 2: u and v share an edge in G

This means they were part of the same connected component in G. Take an arbitrary vertex x in a different connected component in G. Edges $u - x$ and $v - x$ must both exist in G^c. Thus, there is a path $u - x - v$ between vertices u and v.

Thus, we have shown that there exists a path between any two arbitrary vertices in G^c. By definition G^c must be connected. The claim is proved.