CIS 1600

Recitation Guide - Week 5

Topics Covered: Strong Induction, Intro to Probability

Problem 1:

A standard 52-card deck consists of cards labelled 2 through 10, an Ace, Jack, Queen and King, each with four suits. A hand consists of five cards drawn from the deck. Max is a wannabe magician who is trying to draw specific hands for his new magic show: The Appearing Pigeon. However, Max can't quite consistently draw a specific hand, but he has learned how to draw any hand uniformly at random from the Val school of magic.

(a) Calculate the probability that he draws a four of a kind successfully. A hand is considered "four of a kind" if it contains all four suits of a specific label.

(b) Calculate the probability that he draws a full house successfully. A hand is considered "full house" if it contains three cards of the same label and two cards of the some other label (i.e. 3 Aces and 2 8s).

Solution:

(a) The sample space, Ω, is all the possible ways in which 5 cards can be chosen from the 52 card deck. $|\Omega| = \binom{52}{5}$. Since we are equally likely to pick any hand from the deck, note that the sample space is uniform.

Let A $\subseteq \Omega$ be the event (set of outcomes) where he draws a 4 of a kind. Since Ω is a uniform sample space, $\Pr[A] = \frac{|A|}{|\Omega|}$.

We can compute $|A|$ as follows:

Step 1: Pick a label. $\binom{13}{1}$ ways

Step 2: Pick the 4 cards having the same label. $\binom{4}{4}$ ways

Step 3: Pick the suit for the 5th card. $\binom{4}{1}$ ways

Step 4: Pick the label of the 5th card. $\binom{12}{1}$ ways (We already picked 4 cards of the same label, there are 12 labels left)

By the Multiplication Rule, $|A| = \binom{13}{1} \times \binom{4}{4} \times \binom{4}{1} \times \binom{12}{1}$. Thus, $\Pr[\text{A}] = \frac{|A|}{|\Omega|} = \frac{\binom{13}{1} \times \binom{4}{4} \times \binom{4}{1} \times \binom{12}{1}}{\binom{52}{5}}$.

(b) Let $A \subseteq \Omega$ be the event (set of outcomes) where he draws a full house. Since Ω is a uniform sample space, $\Pr[A] = \frac{|A|}{|\Omega|}$.

We can compute $|A|$ as follows:

Step 1: Pick a label for the three of a kind. $\binom{13}{1}$ ways

Step 2: Pick the suits in that triple. $\binom{4}{3}$ ways

Step 3: Pick a label for the pair. $\binom{12}{1}$ ways

Step 4: Pick the suits for the pair. $\binom{4}{2}$ ways

By the Multiplication Rule, $|A| = \binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}$. Thus $\Pr[A] = \frac{|A|}{|\Omega|} = \frac{\binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}}{\binom{52}{5}}$.

Problem 2:

Prove for an arbitrary positive integer a that for all positive integers n that $a^n - 1$ is divisible by $a - 1$.

Solution:

We will prove the claim using induction on n.

Define $P(n)$ to be the claim that $a^n - 1$ is divisble by $a - 1$.

Base Case: $n = 1$.

$a^1 - 1 = a - 1$

Since $a - 1$ is clearly divisible by itself the base case holds.

Induction Step: Let k be an arbitrary integer such that $k \geq 1$.

Induction Hypothesis: Assume $P(k)$ is true, that is $a^k - 1$ is divisible by $a - 1$

We want to show the claim is true for $k + 1$, that is that $a^{k+1} - 1$ is divisible by $a - 1$.

Start by considering $a^{k+1} - 1$.

$= a^{k+1} - a + a - 1$

$= a(a^k - 1) + a - 1$

By our IH we know that $a^k - 1$ is divisible by $a - 1$. Therefore we can rewrite $a^k - 1$ as $c(a - 1)$ where c is some integer.

$= a \cdot c(a - 1) + a - 1$

$= (ac + 1)(a - 1)$.

Since a and c are both integers, $ac + 1$ is also an integer. Therefore a^{k+1} is divisible by $a - 1$ since it equals $(ac + 1)(a - 1)$.

Problem 3: Compute the probability of the event "when we roll n (distinguishable) fair dice, any k of the dice show a 4 while the other $n-k$ do not show a 4". Assume $0 \le k \le n$.

Solution:

As discussed in class, we have a uniform probability space whose outcomes are sequences of length n of numbers from [1..6]. In other words, the Ω is given by the Cartesian product of $[1..6] \times \cdots \times [1..6]$ (n times), i.e., $\Omega = [1..6]^n$. By the Multiplication Rule, there are $6 \times \cdots \times 6 = 6^n$ such sequences so $|\Omega| = 6^n$.

Let A be the event where we get exactly k fours.

Next we will find $|A|$.

First we choose k of the n positions to be 4s. $\binom{n}{k}$ ways.

Next we for each of the remaining $n-k$ spots we choose a roll that is not a four. 5^{n-k} ways.

By the multiplication rule,

$$|A| = \binom{n}{k} 5^{n-k}$$

Since the sample space is uniform,

$$Pr(A) = \frac{|A|}{|\Omega|} = \binom{n}{k} \frac{5^{n-k}}{6^n}$$