CIS 1600

Recitation Guide - Week 4

Topics Covered: Induction, Pigeonhole Principle, Binomial Theorem

Problem 1:

Suppose we have the following sequence:

$$a_1 = 1 \qquad\qquad a_2 = 3 \qquad\qquad a_i = a_{i-2} + 2a_{i-1}, \quad i \in \mathbb{Z}, i \geq 3$$

Use induction to prove that for all integers $n \geq 1$, a_n is odd.

Solution:

We see that we need to use strong induction for this problem since our Induction Step involves both the kth and $k-1$th element.

Define $P(k)$ to be the claim that a_k is odd.

<u>Induction Hypothesis:</u> Assume $P(j)$ is true, $\forall j$ s.t. $1 \leq j \leq k$, for some $k \in \mathbb{Z}$, $k \geq 2$

<u>Base Case 1:</u> When $n = 1$, $a_1 = 1$, which is odd. ✓

<u>Base Case 2:</u> When $n = 2$, $a_2 = 3$, which is also odd. ✓

<u>Induction Step:</u> Let k be arbitrary, $k \in \mathbb{Z}, k \geq 2$.

We want to show that $P(k+1)$ is true. We know $a_{k+1} = a_{k-1} + 2a_k$. By the Induction Hypothesis, a_{k-1} and a_k are both odd. Let $a_{k-1} = 2x + 1$ and $a_k = 2y + 1$, for some integers x and y. Hence,

$$\begin{aligned} a_{k+1} &= (2x+1) + 2(2y+1) \\ &= 2x + 1 + 4y + 2 \\ &= 2(x + 2y + 1) + 1 \end{aligned}$$

Which is odd because $x + 2y + 1 \in \mathbb{Z}$

Problem 2: Let S be a set of 16 distinct positive integers such that $\forall x \in S$, $x < 60$. Show that there exists distinct integers $a, b, c, d \in S$ such that $a + b = c + d$.

Solution:

We will prove this using the Pigeonhole Principle.

Every pair of integers will have an associated sum, and there are $\binom{16}{2} = 120$ unordered pairs of distinct elements in S. Since all elements of S are between 1 and 59 inclusive, the sum of any pair of distinct elements will be between 3 and 117 inclusive, which gives 115 possibilities.

Let the unordered pairs represent the pigeons and the possible sums represent the holes. Since there are 120 pigeons and 115 holes, by PHP there exist $\lceil 120/115 \rceil = 2$ distinct pairings that map to the same sum.

However, we are not quite done yet. What if the unordered pairs overlap? If the 2 inputs that map to the same sum are $\{a, b\}$ and $\{a, c\}$ (with distinct a, b, c), then this would be invalid. This would imply, however, that $a + b = a + c \implies b = c$, which contradicts the fact that a, b, c are distinct. Thus, the two pairings that have the same sum have no overlaps.

Problem 3: Our favorite head TAs Andrew and Eric are playing a game in which there are two non-empty bags with an equal number of marbles in them. In this game, the two players take turns removing marbles from one of the bags. In each turn, the player can remove any positive number of marbles as long as they are all from the same bag. The winner of the game is the player that removes the last marble. In Andrew and Eric's current configuration, both bags initially start with the same number of marbles. Prove that one of them can guarantee to always win.

Solution:

Consider the following strategy: the player who goes second always removes the same number of marbles as the player who went first, but from the other bag. If Andrew goes first, Eric can always win by using this strategy.

Define $P(n)$ to be the claim that this strategy always works for bags that start with n marbles each. We prove our strategy works by strong induction.

Induction Hypothesis: Assume $P(j)$ is true, for $1 \leq j \leq k$, for some $k \in \mathbb{Z}^+$.

Base Case: $k = 1$. Andrew goes first. His only move is to remove one marble from a bag. Eric then removes the last marble from the other bag. Thus the strategy works.

Induction Step: We want to show that the claim still holds if each bag has $k + 1$ marbles. So, we start with two bags containing $k + 1$ marbles each. In Andrew's first move, he can remove m number of marbles for $m \in \mathbb{Z}$, $1 \leq m \leq k + 1$.

Case 1: $m = k + 1$ (i.e., Andrew removes all the marbles from a bag).

In this case, Eric can just take the $k+1$ marbles in the other bag. Because he took the last marble, he wins.

Case 2: $1 \leq m \leq k$:

Thus, after the first move, the bags contain $k+1-m$ and $k+1$ marbles. According to the strategy, Eric removes m marbles from the other bag so that both bags now contain $k + 1 - m$ marbles. We can view the current state of the game as a new game in which both piles contain $k+1-m$ marbles. Since $1 \leq k + 1 - m \leq k$, we can apply the Induction Hypothesis to state that this strategy will always work.

Problem 4:

All the sheep in Winston's flock have the same color! Winston claims that he can use induction to prove all sheep in the world have the same color. Find the fault in his reasoning.

<u>Base Case:</u> size = 1. One sheep, one color. ✓

<u>Induction Hypothesis:</u> Assume that in a flock of size k, where $k \in \mathbb{Z}^+$, all sheep have the same color.

<u>Induction Step:</u> We want to prove the claim is true for a flock of size $k+1$. Take one sheep, let's call her Sara, out. What remains is a flock of size k, so by IH, they all share the same color. Now put Sara back in and take out another sheep, let's call him Thomas Zeuthen, out. By IH, what remains is a flock of size k, so by IH, they all share the same color. Sara and Thomas Zeuthen must share the same color as they both are the same color as the other $k-1$ sheep. Thus we arrive at the conclusion that all $k+1$ sheep share the same color.

Solution:

The problem lies in the step where we try to show that having one sheep of the same color implies that two sheep must have the same color (That is, when $k = 1$ and $k + 1 = 2$). It is possible that the sheep don't share the same color. Note that in the induction step when we remove one of the sheep (say Sara), Thomas Zeuthen would be the only sheep left behind since there are $k - 1 = 0$ sheep left over. Thus, we can only conclude that Thomas Zeuthen has the same color as himself. Similarly, after the other removal, we would only be able to conclude the Sara is the same color as herself. Thus, we can't say anything about Thomas Zeuthen or Sara sharing the same color as another sheep, so our logic for the induction step does not work.

Thus, the induction used in the question is not valid.

(Note: For this proof, we may have wanted to attempt two bases to avoid this problem (i.e. proving when $n = 2$ that we are good), but as shown above, the claim is not necessarily true in this example, so the claim is false.)