CIS 160

Recitation Guide - Week 3

Topics Covered: Combinatorial Proofs, Multisets, PIE

Problem 1:

Given numbers 1 to 9, how many permutations of the numbers do not have at least 7 consecutively increasing numbers? Note that the sequence 1, 2, 3 is consecutively increasing, while 1, 4, 6 is not.

Solution:

We solve this problem with complementary counting, a way of counting used when solving problems where the number of allowed cases is much harder to enumerate than the number of disallowed cases. First, the total number of permutations without restrictions are 9!. Now we need to subtract from that number the permutations which give at least 7 consecutively increasing numbers. Note that any sequence of consecutive numbers can only begin from the first, second or third position. Therefore, there are three sets we need to consider:

$C_1 =$ The set of sequences with at least 7 consecutive numbers beginning from the first position in line.

$C_2 =$ The set of sequences with at least 7 consecutive numbers beginning from the second position in line.

$C_3 =$ The set of sequences with at least 7 consecutive numbers beginning from the third position in line.

The number of sequences that cannot be included is then represented by $|C_1 \cup C_2 \cup C_3|$. By the principle of inclusion-exclusion (PIE), we can compute this as

$$|C_1 \cup C_2 \cup C_3| = |C_1| + |C_2| + |C_3| - |C_1 \cap C_2| - |C_1 \cap C_3| - |C_2 \cap C_3| + |C_1 \cap C_2 \cap C_3|$$

The cardinality $|C_1|$ can be computed using the multiplication rule:

> Step 1: Select the number at the first position: The only options for the first number are 1, 2, and 3, so this can be done in 3 ways.
>
> Step 2: Select the next six numbers: Since the numbers are consecutively increasing, this can be done in 1 way.
>
> Step 3: Select the eighth number: There are two numbers available at this point, so it can be done in 2 ways.
>
> Step 4: Select the last number: With only one number remaining, this can be done in 1 way.

Therefore, $|C_1| = 3 \times 1 \times 2 \times 1 = 6$.

Similarly, the cardinality of C_2 and C_3 are also 6.

Now, we must calculate the cardinalities of the intersections of sets:

$|C_1 \cap C_2|$ can also be calculated using the multiplication rule:

> Step 1: Pick the first number. This can be done in two ways, as the first number can only be 1 or 2.
>
> Step 2: Pick the next 7 numbers. Because the sequence is consecutively increasing from 1-7 and from 2-8, there is only one way to do this.
>
> Step 3: Pick the last number. There is only one number remaining, so this can be done in one way.

Thus, $|C_1 \cap C_2| = 2$. Similarly, we have $|C_2 \cap C_3| = 2$.

For $|C_1 \cap C_3|$, note that this can only be the sequence $1, 2, 3, \ldots, 9$. The same is true for $|C_1 \cap C_2 \cap C_3|$.

Putting this all together, we get the following expression:

$$\begin{aligned}
|C_1 \cup C_2 \cup C_3| &= |C_1| + |C_2| + |C_3| - |C_1 \cap C_2| - |C_1 \cap C_3| - |C_2 \cap C_3| + |C_1 \cap C_2 \cap C_3| \\
&= 6 + 6 + 6 - 2 - 2 - 1 + 1 \\
&= 14
\end{aligned}$$

Finally we can deduct this number from the total number of permutations without restrictions, making the answer

$$\boxed{9! - 14}$$

Problem 2:

Give a combinatorial proof for the following:

$$2^n - 1 = \sum_{k=0}^{n-1} 2^k$$

Solution:

Consider the following counting problem:

> How many strings made of 0's and 1's with length n are there such that the string has at least one 1?

First, let's start with the more straightforward side of the equation (this is almost always the side without the summation).

<u>LHS:</u> Count all possible strings of bits of length n and subtract the number of strings in which all bits are 0s. This gives the LHS, $2^n - 1$.

<u>RHS:</u> We count by first considering the location of the last 1 in the string. There is at least one 1 in the string, so we know that the "last" 1 must exist. Thus, once we assign the location of the last 1 bit, every bit before it can be either a 0 or a 1 (and all bits following must be a 0).

We know that the last 1 bit can be anywhere from position $k = 1$ to $k = n$. If the last non-zero bit is in position k, there are 2^{k-1} possible strings which could precede it for each possible case of k.

We can sum over all values of k, since no two strings with different positions of the last 1 could be identical (cases are disjoint).

Thus, we have that the total number of such strings is

$$2^0 + 2^1 + \cdots + 2^{n-1} = \sum_{k=0}^{n-1} 2^k$$

Problem 3: Cindy is proctoring a makeup exam and she needs to distribute scratch paper and pens to students. She starts with 10 sheets of scratch paper and 10 pens. She begins handing out the paper and pens to the students, but after the 6th student, Cindy discovers that she has run out of supplies. Most importantly, she does not remember when her supplies ran out (meaning she could have given all of her supplies to the first student). She cannot tell the difference between any two sheets of scratch paper and between any two pens. However, she can easily tell the difference between a sheet of scratch paper and a pen.

How many ways could Cindy have distributed the scratch paper and pens to the different students?

Solution:

We can break this problem down into separate sticks and crosses problems and combine them at the end.

There are 6 students in which we distribute 10 sheets of scratch paper. Arrange 10 crosses in a row. These crosses represent the sheets of scratch paper. Since the papers are indistinguishable, their ordering is irrelevant. We now also have $6 - 1 = 5$ sticks to represent the students. Place the 5 sticks between some of the crosses (scratch papers). The sticks would then separate the crosses into 6 parts, each of which represent one student. The number of arrangements of sheets of scratch paper is then

$$\binom{10+6-1}{10} = \binom{15}{10}$$

We can do the same for the pens. 10 indistinguishable pens distributed to 6 students gives us

$$\binom{10+6-1}{10} = \binom{15}{10}$$

But this isn't the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the scratch paper is distributed is independent of the order in which the pens are distributed.

Step 1: Choose a way to distribute the scratch paper. From above, there are $\binom{15}{10}$ ways.

Step 2: Choose a way to distribute the pen. Again, there are $\binom{15}{10}$ ways.

This gives us

$$\binom{15}{10} \times \binom{15}{10} = \boxed{\binom{15}{10}^2}$$