CIS 1600

Recitation Guide - Week 2

Topics Covered: Proofs, Counting

Problem 1: Let m and n be two integers. Prove that $mn + m$ is odd if and only if m is odd and n is even.

Solution:

Lemma 1: For two integers x and y, if xy is odd, then x and y are both odd.

We will prove the Lemma through a proof by contrapositive. In other words, we will prove "If x or y is even, then xy is even."

WLOG, let x be the even integer.

We can write $x = 2k$, for some $k \in \mathbb{Z}$. Then, we have,

$$\begin{aligned} xy &= (2k)(y) \\ &= 2(ky) \end{aligned}$$

which is even, as $ky \in \mathbb{Z}$.

($\Longrightarrow$) If $mn + m$ is odd, then m is odd and n is even.

We can write $mn + m = m(n + 1)$. Then, according to Lemma 1, m and $n + 1$ must both be odd.

Since we know that $n + 1$ is odd and 1 is odd, then n must be an even integer (because only even + odd = odd).

Therefore, m is odd and n is even.

($\Longleftarrow$) If m is odd and n is even, then $mn + m$ is odd.

We can write $m = 2k + 1$, for some $k \in \mathbb{Z}$ and $n = 2l$, for some $l \in \mathbb{Z}$. Then, we have,

$$\begin{aligned} mn + m &= (2k + 1)(2l) + (2k + 1) \\ &= 4kl + 2l + 2k + 1 \\ &= 2(2kl + l + k) + 1 \end{aligned}$$

which is odd, as $2kl + l + k \in \mathbb{Z}$.

Problem 2: Let $A = \{n \,|\, n = 2k + 5 \text{ for some } k \in \mathbb{N}\}$ and $B = \{n \,|\, n = 2j + 1 \text{ for some } j \in \mathbb{N}\}$. Is $A \subseteq B$?

Solution:

Let x be any arbitrary but particular element in A. Then,

$$\begin{aligned} x &= 2k + 5, \quad \text{for some integer } k. \\ &= 2(k + 2) + 1 \end{aligned}$$

Since $k \in \mathbb{N}$, $k + 2 \in \mathbb{N}$, and hence we have proved that any arbitrary element $x \in A$ also belongs to the set B. Thus $A \subseteq B$.

Problem 3:

Count the number of sequences of bits of length 5 in which every 0 is followed immediately by a 1.

Solution:

Observe that we can have at most two 0's in the sequence. Otherwise, at least one 0 will not be followed immediately by a 1. Thus, we can case on the number of 0's in the final sequence.

Case 1: The sequence doesn't contain any 0's. There is only one such sequence: 11111, so there is 1 such sequence in Case 1.

Case 2: The sequence contains one 0 and four 1's.

Put all 1's in a line, and we can insert the only 0 in front of any 1's. Since there are 4 choices, the number of sequences is $\binom{4}{1} = 4$.

In other words, using the Multiplication Rule:

> *Step 1:* Arrange four 1's in a line. (1 way)
>
> *Step 2:* Choose one position to place 0. ($\binom{4}{1}$ ways)

Since the number of ways to do each steps are independent, by the Multiplication Rule, there are $1 \times \binom{4}{1} = 4$ such sequences in Case 2.

Case 3: The sequence contains two 0's and three 1's.

Again, put all 1's in a line. Think of spaces in front of the first 1 and in between each 1 as available spots.

We can place at most one 0 into each spot. Otherwise, two 0's would be next to each other and one of them will not be followed immediately by a 1. Using this method, we can choose 2 spots from the 3 available spots, and place a zero in each one.

In other words, using the Multiplication Rule:

> *Step 1:* Arrange three 1's in a line. (1 way)
>
> *Step 2:* Choose two positions to place 0. ($\binom{3}{2}$ ways)

Since the number of ways to do each steps are independent, by the Multiplication Rule, there are $1 \times \binom{3}{2} = 3$ such sequences in Case 3.

Case 3: (Alternative Approach) The sequence contains two 0's.

Using Multiplication Rule:

> *Step 1:* Choose 2 locations from the 4 available spots to place the 0. Note that the 0 cannot be in the 5th location, because it can't be followed immediately by 1.
>
> We can find the total ways to place two 0's using $\binom{4}{2}$, then subtract the ways in which the 0's are next to each other (because then there would not be a 1 after the first 0).

There are 3 ways in which the two 0's would be placed consecutively, so there are $\binom{4}{2}$ - 3 = 3 ways to place the 0's with 1's after.

Step 2: Fill the remaining spots with 1's. (1 way)

Since the number of ways to do each steps are independent, by the Multiplication Rule, there are $(\binom{4}{2} - 3) \times 1 = 3$ such sequences in Case 3.

Since all 3 cases are disjoint, we can use the addition rule to get the total number of desired sequences. Therefore, the total number of sequences in which every 0 is followed immediately by a 1 is

$$1 + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8$$