CIS 1600 Recitation 2

Counting, Sets, Proof Techniques

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Sets

- A *set* is an unordered collection of distinct objects. $\{a, b, c\}$
- Two sets are *equal* if and only if they have the same elements.
- The *cardinality* $|S|$ is the number of elements in S.
- A is a *subset* of B ($A \subseteq B$) if and only if every element of A is an element of B.
- A *powerset* of S ($\mathcal{P}(S)$) is a set of all possible subsets of S.
- $\mathbb{N} = \{0, 1, 2, 3, ...\}, \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$,

  $\mathbb{Q} =\{\text{p/q} |\ p \in \mathbb{Z} \land q \in \mathbb{Z} \land q \neq 0\}, \mathbb{R}$ is a set of real numbers.

Set Operations

- *Union*: x is in $A \cup B$ if x is in A, B, or both
- *Intersection*: x is in $A \cap B$ if x is in both A and B
- *Disjoint*: A, B are disjoint if $A \cap B$ is the empty set.
- *Partition*: $\{A_1, A_2, ..., A_n\}$ is a partition of A if and only if (i) $A = \bigcup_{i=1}^{n} A_i$ and (ii) $A_1, A_2, ..., A_n$ are mutually (pairwise) disjoint.
- *Difference*: $A \setminus B$ or $A - B$ contains elements that are in A not but in B.
- *Complement*: $\bar{A}$ contains all elements not in A. $\bar{A} = U \setminus A$ when U is the universe of elements.
- *Cartesian Product*: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

DeMorgan's Laws

- Let A, B, C be sets.

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

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Permutations

Let A be a non-empty set with n elements, that is, $|A| = n$.

- A **permutation** of A is an ordering of the elements of A in a row *without repetition*.
- The number of permutations is $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$.
- A **partial permutation** of r out of the n elements of A consists of picking r of the elements of the set and ordering them in a row without repetition.
- The number of partial permutations is $\frac{n!}{(n-r)!}$.

Combinations

Let A be a non-empty set with n elements, that is, $|A| = n$.

- A **combination** of r elements from the n elements of A is an *unordered* selection of r of the n elements of A, where r is a natural number.
- The number of combinations is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$
- $\binom{n}{r} = \begin{cases} 0, \text{if } r > n \\ 1, \text{if } r = 0 \text{ or } r = n \\ n, \text{if } r = 1 \end{cases}$

Proof by Contradiction

- To prove that a claim p is true, assume that $\neg p$ is true. Then, we have to show that $\neg p$ is false by reaching a contradiction (some statement that is always false).
- For example, if we want to prove that $p \implies q$, we could show that $p \wedge \neg q$ leads to a false statement.
- **Example:** Prove that "if $3n + 2$ is odd then n is odd." We can show this by contradiction by assuming that "$3n + 2$ is odd and n is even" and showing that this statement is false. (Proof done in Lecture 2H)

Proof by Contrapositive

- To prove that $p \implies q$ is true, we can show that $\neg q \implies \neg p$ is true.
- **Example:** Prove that "if x and y are integers where $x + y$ is even, then x and y are both odd or both even". We can show this by proving the contrapositive, "if exactly one of x or y is even then $x + y$ is odd". (Proof done in Lecture 2H)
- Similar to proof by contradiction but in proof by contradiction you assume both p and $\neg q$ are true where as in proof by contrapositive you just assume $\neg q$ is true.