CIS 1600

Recitation Guide - Week 13

Topics Covered: Planar Graphs, Functions

Problem 1: Prove that in any connected planar graph with minimum degree 3, there are at least $\frac{n}{2} + 2$ faces

Solution:

Euler's Formula tells us that the number of faces in a connected planar graph is $f = 2 - n + m$. From Handshaking Lemma, we know that $m \geq \frac{3n}{2}$. This gives the following:

$$\begin{aligned} f &= 2 - n + m \\ &\geq 2 - n + \frac{3n}{2} \\ &= 2 + \frac{n}{2} \end{aligned}$$

Problem 2: Let $m, n \geq 2$. Define:

$$f : [1..m] \times [1..n] \to [2..(m+n)] \text{ by } f(x,y) = x + y.$$

Is f an injection? Is f a surjection? Is f a bijection? Prove your answers.

Solution:

- f is not injective. In order to show this, we will provide two elements $(a,b),(c,d) \in [1..m] \times [1..n]$ such that $(a,b) \neq (c,d)$ but $f(a,b) = f(c,d)$. To this end, consider $(1,2)$ and $(2,1)$. Clearly, $(1,2) \neq (2,1)$. However, since $f(1,2) = 3 = f(2,1)$, they both map onto the same element in the codomain, and we conclude that f cannot be injective.

- f is surjective. To prove surjectivity, we must show that $\forall z \in [2..(m+n)], \exists x \in [1..m], y \in [1..n]$ such that $f(x,y) = z$. Intuitively, we want to show that everything in the codomain of f is mapped to by at least one element from the domain. One way to do this is to consider any z in the codomain, i.e., any $z \in [2..(m+n)]$, and consider the following cases:

  Case 1: $2 \leq z \leq m+1$.

  In this case, let $y = 1$. Clearly, $1 \in [1..n]$. Then we need $x = z - 1$, by definition of f. Substituting into the fact that $2 \leq z \leq m+1$, we see that $1 \leq x \leq m$, meaning x will always be an element of $[1..m]$. Thus, we know that $(x,1) \in [1..m] \times [1..n]$, and we have found an element of the domain, namely $(x,1)$, such that $f(x,1) = z$.

  Case 2: $m+1 < z \leq m+n$.

  In this case, let $x = m$. Clearly, $m \in [1..m]$. Then we need $y = z - m$, by definition of f. As above, by substituting into the fact that $m+1 < z \leq m+n$, we see that $1 < y \leq n$, meaning y will always be an element of $[1..n]$. As above, we know that $(m,y) \in [1..m] \times [1..n]$, and we have found an element of the domain, namely (m,y), such that $f(m,y) = z$.

  In both cases, we have found an element of the domain (x,y) such that $f(x,y) = z$, so we know that f is surjective.

- f is not bijective because a bijection must be both injective and surjective, and f is not injective.