CIS 1600

Recitation Guide - Week 12

Topics Covered: Probabilistic Method

Problem 1:

Let G be a bipartite graph with $|V| = n$. Suppose you give each vertex its own list of more than $\log_2 n$ possible colors.

Show that it is possible to provide a valid coloring of G choosing each vertex's color from its own list

Solution:

Since G is bipartite we can partition the vertices into two set V_1 and V_2 such that all of the edges go between one vertex in V_1 and one vertex in V_2.

Let X be the total set of colors from all lists. For each color in X we can flip a fair coin, so that our sample space $\Omega = \{(c_1, \ldots, c_{|X|}) \mid c_i \in \{H, T\}\}$. When assigning colors the vertices in V_1 can only take colors that got heads, and the vertices in V_2 can only take colors that got tails. This will result in a valid coloring since all the vertices, since a vertex in V_1 can never have the same color as a vertex in V_2.

We want to find the probability that that this strategy fails, that is that there exists some vertex in V_1 with all colors in its list getting tails or a vertex in V_2 with all colors in its list getting heads.

Let N be a random variable representing the number of vertices for which this process fails.

Let N_i be an indicator that is one if the ith vertex fails, 0 otherwise.

Suppose the length of the list of colors of vertex i is k.

$P(N_i = 1) = \frac{1}{2^k}$ since this is the probability every one of N_i's colors in its list independently got the wrong color.

Since $k > \log_2 n$, $P(N_i = 1) < \frac{1}{n}$, so $\mathbb{E}(N_i) < \frac{1}{n}$.

Since there are n vertices in the graph, we know that $N = \sum_{i=1}^{n} N_i$.

By the linearity of expectation, $\mathbb{E}(N) = \sum_{i=1}^{n} \mathbb{E}(N_i) < 1$.

Since $\mathbb{E}(N) < 1$, there exists a way to coloring that fails for no vertices.

Problem 2:

Given an arbitrary graph $G(V, E)$, show that there exists an independent set of size at least:

$$\sum_{v \in V} \frac{1}{\deg(v) + 1}$$

Solution:

We proceed by using the probabilistic method. We must come up with a random procedure to select a subset of the vertices such that they make up an independent set.

Consider a permutation of all the vertices uniformly at random, call it π. Thus, our sample space Ω consists of all possible permutations of vertices. Consider a subset of vertices W, that appear before each of their neighbors in π. That is where for each $w \in W$, if there is an edge between w and some other vertex u, w appears before u in π.

First, we note that W must be an independent set. Consider an arbitrary $w \in W$. For each of its neighbors v, we know that $v \notin W$, as w is a neighbor of v which appears earlier in π; thus w will not be adjacent to any other vertex in W.

Now, let us define an random variable X denoting the size of W. We define indicators X_i which equals 1 if vertex v_i is in W, and 0 otherwise. We note that

$$X = \sum_{i=1}^{|V|} X_i$$

As such, we apply linearity of expectation to get that

$$\mathbf{E}[X] = \sum_{i=1}^{|V|} \mathbf{E}[X_i]$$

Now, it remains to calculate $\mathbf{E}[X_i]$. We note that as the permutations are chosen uniformly at random, the probability that v_i appears first before all of its neighbors is $\frac{1}{\deg(v_i)+1}$

Since every permutation of all the vertices are equally likely to be selected, every permutation of v_i and its neighbors is also equally likely to be selected. Since only v_i and its neighbors affect $P(X_i = 1)$ we can just consider the $\deg(v_i)+1$ vertices that make up v_i and its neighbors. There are $(\deg(v_i) + 1)!$ total permutations of these vertices. Additionally, there are $\deg(v_1)!$ permutations where v_i appears first since you first give v_i the first spot then you arrange the other $\deg(v_i)$ vertices.

This gives us a probability of $\frac{\deg(v_i)!}{(\deg(v_i)+1)!} = \frac{1}{\deg(v_i)+1}$.

As such, we see that

$$\mathbf{E}[X] = \sum_{i=1}^{|V|} \frac{1}{\deg(v_i) + 1}$$

Therefore, we can then conclude that there must exist some permutation π yielding W with cardinality at least $\mathbf{E}[X]$, as desired.