CIS 1600

Recitation Guide - Week 11

Topics Covered: Relations, Hall's Theorem

Problem 1:
Consider a normal chessboard (an 8×8 grid). In each row and in each column there are exactly n pieces, where $0 < n \leq 8$. Prove that we can pick 8 pieces such that no two of them are in the same row or column.

Solution:

We construct a bipartite graph G as follows. Let X be the set of rows modeled as vertices. Let Y be the set of columns modeled as vertices. Let E be the set of edges such that if a piece exists in row i and column j, then there is an edge between $x_i \in X$ and $y_j \in Y$. Note that the graph must be bipartite because no edges exist between two vertices in X or two vertices in Y.

The question asks us to find a matching: can we match each of the 8 rows to a unique column? Note that this would mean that we could pick 8 edges (in our matching) that are not in the same row or same column.

We must prove the existence of such a perfect matching. First, note that the size of our two bipartite sets X and Y are the same since there are exactly 8 rows and 8 columns; in other words, $|X| = |Y| = 8$. Hence, if we can find a matching that saturates X, then it must also saturate Y (and so is a perfect matching). To prove the existence of this matching, we show that Hall's Condition is satisfied, that is that $|N_G(S)| \geq |S|, \forall S \subseteq X$.

Consider an arbitrary but particular subset $A \subseteq X$ (of the rows). Recall that there are n pieces in each row and n pieces in each column. Thus, there must be $n|A|$ edges from A to $N_G(A)$. We also know that each column in $N_G(A)$ has at most n edges back to A, meaning that there are at most $n|N_G(A)|$ edges from $N_G(A)$ to A. This means that $n|A| \leq n|N_G(A)|$, meaning that $|A| \leq |N_G(A)|$. This satisfies Hall's Condition, leading us to prove the existence of our matching.

Problem 2:
Define an equivalence relation R on the set $\{1, 2, 3, \ldots, 100\}$ with the restriction that there are exactly 2 equivalence classes. Find an R such that it maximizes the size of the relation, and then show that the size is maximized.

Solution:

WLOG, let us label the two equivalence classes A and B, such that $|A| = a$ and $|B| = 100 - a$. We know that since R is an equivalence relation, $R = (A \times A) \cup (B \times B)$.

Hence $|R| = |A| \cdot |A| + |B| \cdot |B| = a^2 + (100 - a)^2 = 2a^2 - 200a + 10000 = 2(a^2 - 100a + 250) + 9500 = 2(a - 50)^2 + 9500$. In order to maximize $|A|$, we seek to maximize $2(a - 50)^2$. Note that the extremal values 1 and 99 maximize it. Hence, we can set $|A| = 99$ and $|B| = 1$ to maximize the cardinality of the equivalence relation.

Problem 3:

Consider a set A with $n \geq 1$ elements. We color independently each of the elements of A red with probability $\frac{1}{3}$ and blue with probability $\frac{2}{3}$. Let R be the "is the same color as" relation on A, ie. if a is the same color as b, then $(a, b) \in R$.

a) Is R an equivalence relation? If so, what are its equivalence classes?

b) Calculate the expected value of $|R|$.

Solution:

a) R is an equivalence relation:

- Reflexive: $\forall a \in A$, a must be the same color as itself, so aRa.
- Symmetric: given aRb, then a must be the same color as b, so b and a must be both blue or both red. Therefore b must be the same color as a, so bRa.
- Transitive: given aRb and bRc, then a must be the same color as b, and b must be the same color as c. Since b only has one color, then a and c must be the same color.

Since every element in an equivalence class defined by R must be related by R, then each element in such an equivalence class has the same color. Therefore, R determines two equivalence classes of A: in one equivalence class we have all the blue elements and in the other equivalence class we have all the red elements. These determine a partition of A based on color.

b) The elements of R are ordered pairs (x, y) where $x, y \in A$. The sample space is all the possible cardinalities of R.

First we examine the case $x \neq y$. For the pair (x, y) to be in R both x and y must be colored with the same color. Using the independence, the probability that they are both red is $(\frac{1}{3})(\frac{1}{3}) = \frac{1}{9}$. Similarly, the probability that they are both blue is $(\frac{2}{3})(\frac{2}{3}) = \frac{4}{9}$. Since these are disjoint, the probability that they are colored with the same color is $(1/9) + (4/9) = 5/9$, so the probability that $(x, y) \in R$ is $5/9$. Note that there are $n(n-1)$ such (x, y) where $x \neq y$.

Next we examine the case $x = y$. In this case, $(x, y) \in R$ must be true, by reflexivity, so the probability that $(x, y) \in R$ is 1. Note that there are n such (x, y) where $x = y$.

Now define for each $(x, y) \in A \times A$ an indicator random variable $X_{x,y}$ that is 1 when $(x, y) \in R$ and 0 otherwise. We have $\mathbb{E}(X_{x,y}) = \Pr[(x, y) \in R]$ which equals $5/9$ when $x \neq y$ and 1 when

$x = y$. By linearity of expectation, where we let the sum range over all $(x, y) \in A \times A$:

$$\begin{aligned}
\mathbb{E}\Big[|R|\Big] &= \mathbb{E}\left[\sum_{x,y} X_{x,y}\right] \\
&= \sum_{x,y} \mathbb{E}[X_{x,y}] \\
&= \sum_{x \neq y} \mathbb{E}[X_{x,y}] + \sum_{x} \mathbb{E}[X_{x,x}] \\
&= \frac{5}{9} \cdot n(n-1) + 1 \cdot n \\
&= \frac{n(5n+4)}{9}
\end{aligned}$$