CIS 1600

Recitation Guide - Week 10

Topics Covered: Memoryless Property, Probability Distributions, Chebyshev's Inequality

Problem 1:

A 10 digit number with no zeroes is chosen by independently and randomly selecting each digit (1 - 9).
Let N be the number of digits missing from the 10 digit number. For example, if the number is 1231452832, then we are missing the digits 6, 7, 9 so $N = 3$.

$$\mathbb{E}[N] = 9 \cdot \left(\frac{8}{9}\right)^{10} \approx 2.772$$

$$\text{Var}[N] = 9\left(\frac{8}{9}\right)^{10} + 72\left(\frac{7}{9}\right)^{10} - 81\left(\frac{8}{9}\right)^{20} \approx 0.9232$$

Using Markov's Inequality, we found the lower bound of the probability that less than 6 digits are missing to be at least

$$1 - \frac{9 \cdot \left(\frac{8}{9}\right)^{10}}{6}$$

.
How can you improve the bound you obtained above?

Solution:

We can use Chebyshev's inequality:

$$\Pr[|N - \mathbb{E}[N]| \geq a] \leq \frac{\text{Var}[N]}{a^2} \qquad \text{(Chebyshev's Inequality)}$$

Our objective should still be to solve for $Pr[N \geq 6]$ using Chebyshev's and then find the complement event. Let us focus on the LHS of Chebyshev's:

$$\begin{aligned}\Pr[|N - \mathbb{E}[N]| \geq a] &= \Pr[N - \mathbb{E}[N] \geq a] + \Pr[-(N - \mathbb{E}[N]) \geq a] \\ &\qquad \text{(since absolute value is piecewise)} \\ &= \Pr[N - \mathbb{E}[N] \geq a] + \Pr[\mathbb{E}[N] - N \geq a] \\ &= \Pr[N \geq a + \mathbb{E}[N]] + \Pr[N \leq \mathbb{E}[N] - a]\end{aligned}$$

Note that by letting $a = (6 - \mathbb{E}[N])$, the first term will result in $\Pr[N \geq 6]$ which is what we are looking for. So let's substitute our chosen value for a into the above:

$$= \Pr[N \geq (6 - \mathbb{E}[N]) + \mathbb{E}[N]] + \Pr[N \leq \mathbb{E}[N] - (6 - \mathbb{E}[N])]$$
$$= \Pr[N \geq 6] + \Pr[N \leq 2\mathbb{E}[N] - 6]$$

.

To recap, we have rewritten $\Pr[|N - \mathbb{E}[N]| \geq a] = \Pr[N \geq 6] + \Pr[N \leq 2\mathbb{E}[N] - 6]$ for a chosen $a = (6 - \mathbb{E}[N])$.

But wowzers! The second term! We have that $2\mathbb{E}[N] - 6 < 0$, and since N is non-negative, it must be that $\Pr[N \leq 2\mathbb{E}[N] - 6] = 0$.

Therefore, $\Pr[|N - \mathbb{E}[N]| \geq a] = \Pr[N \geq 6]$ for $a = (6 - \mathbb{E}[N])$. Now, we finally use Chebyshev's:

$$\begin{aligned}
\Pr[N \geq 6] = \Pr[|N - \mathbb{E}[N]| \geq a] &\leq \frac{\text{Var}[N]}{a^2} \\
&\leq \frac{\text{Var}[N]}{(6 - 9 \cdot (\frac{8}{9})^{10})^2} \\
&\leq \frac{9\left(\frac{8}{9}\right)^{10} + 72\left(\frac{7}{9}\right)^{10} - 81\left(\frac{8}{9}\right)^{20}}{(6 - 9 \cdot (\frac{8}{9})^{10})^2} \\
&\leq 0.0886
\end{aligned}$$

.

Thus, if $\Pr[N \geq 6] \leq 0.0886$, we have $\Pr[N < 6] = 1 - \Pr[N \geq 6] \geq 1 - 0.0886 = 0.9114$

Problem 2:
You are at an auction for a box of money. The amount of money in the box is unknown to you, and is secretly determined by Aaron. Aaron flips a biased coin 100 times, with a 1/3 chance of getting heads, and for each heads that appears, he puts a dollar in the box.

There is only one other bidder at the auction, Elisa, who rolls a 6-sided fair die until she gets a 6, and for each roll adds \$5 to her bid.

Everyone who attends the auction reveals their bid at the same time, and the person with the highest bid pays that amount of money to get the box. (Assume you can only bet in whole dollar amounts.)

a) Let's say you want to bid strictly more than the expected value of Elisa's bid (so you win the box), but strictly less than the expected value of the box (so you still make money). Is that possible?

b) What if Elisa bids according to a 7-sided die, rolling until she gets a 7? In expectation and using the same strategy as a), can you still make money?

Solution:

Note that the sample space is the set of all ordered pairs (x, y) where x represents the sequence of 100 biased coin flips and y represents the sequence of 6-sided die rolls ending with the first 6. That is, $\Omega = \Omega_M \times \Omega_B$, where Ω_M is the sample space for the biased coin flips and Ω_B is the sample space for Elisa's die rolls.

a) Let M be the amount of money in the box, and let B be the amount that Elisa bids. We need to calculate the **E**[M] and **E**[B] bid.

   Note that M is a binomial random variable with $n = 100$ and $p = \frac{1}{3}$. Therefore:

$$\begin{aligned}\mathbb{E}[M] &= np \\ &= \frac{100}{3} \approx 33.33\end{aligned}$$

   Let $B = 5B_6$, where B_6 is the random variable denoting the number of rolls of the 6-sided die up to and including the first roll of 6. Note that B_6 is a geometric random variable with $p = \frac{1}{6}$. By the Linearity of Expectation:

$$\begin{aligned}\mathbb{E}[B] &= \mathbb{E}[5B_6] \\ &= 5\mathbb{E}[B_6] \\ &= 5 \cdot \frac{1}{p} \\ &= 5 \cdot 6 = 30\end{aligned}$$

   Since $30 < 33.33$ and there is a whole dollar amount between the two, we can bid an amount between \$30 and \$33.33 (exclusive) and expect to make money.

b) $\mathbb{E}[M]$ is the same as before, but $\mathbb{E}[B]$ will be different (and the sample space should y represents the sequence of 7-sided die rolls ending with the first 7).

   Now $B = 5B_7$, where B_7 is the random variable denoting the number of rolls of the 7-sided die up to and including the first roll of 7. Note that B_7 is a geometric random variable with

$p = \frac{1}{7}$. By the Linearity of Expectation:

$$\begin{aligned}
\mathbb{E}[B] &= \mathbb{E}[5B_7] \\
&= 5\mathbb{E}[B_7] \\
&= 5 \cdot \frac{1}{p} \\
&= 5 \cdot 7 = 35
\end{aligned}$$

Since $35 > 33.33$, then there is no amount of money we can bid where we can expect to make money according to this strategy.

Problem 3:

For a geometric random variable X with parameter p, where $n > 0$ and $k \geq 0$, we have the memoryless property

$$\Pr[X = n + k \mid X > k] = \Pr[X = n]$$

The following is the definition of conditional expectation.

$$\mathbb{E}[Y \mid Z = z] = \sum_y y \cdot \Pr[Y = y \mid Z = z],$$

a) Prove the law of total expectation below. Given any random variables X, Y, defined in the same sample space,

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y = y] \Pr[Y = y]$$

b) Calculate the expectation of a geometric random variable with the memoryless property and the law of total expectation.

Solution:

a) We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_x x \cdot \Pr[X = x] \\
&= \sum_x x \cdot \sum_y (\Pr[X = x|Y = y] \cdot \Pr[Y = y]) && \text{(By Law of Total Probability)} \\
&= \sum_y \Pr[Y = y] \cdot \sum_x (x \cdot \Pr[X = x|Y = y]) \\
&= \sum_y \Pr[Y = y] \cdot \mathbb{E}[X|Y = y]
\end{aligned}$$

b) We calculate the expectation of a geometric random variable X with parameter p as follows. Seeing as we have the memoryless property, we condition X on the result of the first trial.

Formally, let Y be the indicator random variable that represents the outcome of the first Bernoulli trial, where $Y = 0$ if the first trial is a failure and $Y = 1$ otherwise. Using the law of total expectation, we have

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X \mid Y = 0] \cdot \Pr[Y = 0] + \mathbb{E}[X \mid Y = 1] \cdot \Pr[Y = 1] \\
&= \mathbb{E}[X \mid Y = 0] \cdot \Pr[Y = 0] + 1 \cdot p
\end{aligned}$$

We see that if the first trial was a success, then expectation of X is 1, as there will be no more trials.

Intuitively, since X is memoryless, if the first trial was a failure, the expected number of trials would just be $\mathbb{E}[X] + 1$. Rigorously, we attempt to use the memoryless property on the first term. We have

$$\begin{aligned}
\mathbb{E}[X \mid Y = 0] &= \sum_{x=1}^{\infty} x \cdot \Pr[X = x \mid Y = 0] \\
&= 1 \cdot \Pr[X = 1 \mid Y = 0] + \sum_{x=2}^{\infty} x \cdot \Pr[X = x \mid Y = 0] && \text{(Splitting the sum)}
\end{aligned}$$

Note that if $Y = 0$, then the first trial is a failure. Then X cannot equal 1, because $X = 1$ means that there was a success on the first trial. Therefore $\Pr[X = 1 \mid Y = 0] = 0$.

Note also that $Y = 0$ if and only if $X > 1$, since we must have gone through more than one trial to obtain a success. Substituting these in, we get:

$$\begin{aligned}
\mathbb{E}[X \mid Y = 0] &= 0 + \sum_{x=2}^{\infty} x \cdot \Pr[X = x \mid X > 1] \\
&= \sum_{x=2}^{\infty} x \cdot \Pr[X = (x-1) + 1 \mid X > 1] \\
&= \sum_{x=2}^{\infty} x \cdot \Pr[X = x - 1] && \text{(By the memoryless property)} \\
&= \sum_{x=1}^{\infty} (x+1) \cdot \Pr[X = x] && \text{(Shifting the lower bound back to 1)} \\
&= \sum_{x=1}^{\infty} x \cdot \Pr[X = x] + \sum_{x=1}^{\infty} \Pr[X = x] \\
&= \mathbb{E}[X] + 1
\end{aligned}$$

Hence, putting everything together, we have

$$\begin{aligned}
\mathbb{E}[X] &= (1-p) \cdot (\mathbb{E}[X] + 1) + p \\
\mathbb{E}[X] &= (1-p) \cdot \mathbb{E}[X] + (1-p) \cdot 1 + p \\
\mathbb{E}[X] - (1-p) \cdot \mathbb{E}[X] &= 1 - p + p \\
\mathbb{E}[X] \cdot [1 - (1-p)] &= 1 \\
\mathbb{E}[X] \cdot (p) &= 1 \\
\mathbb{E}[X] &= \frac{1}{p}
\end{aligned}$$