CIS 1600

Recitation Guide - Week 10

Topics Covered: Memoryless Property, Probability Distributions, Chebyshev's Inequality

Problem 1:

A 10 digit number with no zeroes is chosen by independently and randomly selecting each digit (1 - 9).
Let N be the number of digits missing from the 10 digit number. For example, if the number is 1231452832, then we are missing the digits 6, 7, 9 so $N = 3$.

$$\mathbb{E}[N] = 9 \cdot \left(\frac{8}{9}\right)^{10} \approx 2.772$$

$$\text{Var}[N] = 9\left(\frac{8}{9}\right)^{10} + 72\left(\frac{7}{9}\right)^{10} - 81\left(\frac{8}{9}\right)^{20} \approx 0.9232$$

Using Markov's Inequality, we found the lower bound of the probability that less than 6 digits are missing to be at least

$$1 - \frac{9 \cdot \left(\frac{8}{9}\right)^{10}}{6}$$

.
How can you improve the bound you obtained above?

Problem 2:
You are at an auction for a box of money. The amount of money in the box is unknown to you, and is secretly determined by Aaron. Aaron flips a biased coin 100 times, with a 1/3 chance of getting heads, and for each heads that appears, he puts a dollar in the box.

There is only one other bidder at the auction, Elisa, who rolls a 6-sided fair die until she gets a 6, and for each roll adds $5 to her bid.

Everyone who attends the auction reveals their bid at the same time, and the person with the highest bid pays that amount of money to get the box. (Assume you can only bet in whole dollar amounts.)

a) Let's say you want to bid strictly more than the expected value of Elisa's bid (so you win the box), but strictly less than the expected value of the box (so you still make money). Is that possible?

b) What if Elisa bids according to a 7-sided die, rolling until she gets a 7? In expectation and using the same strategy as a), can you still make money?

Problem 3:

For a geometric random variable X with parameter p, where $n > 0$ and $k \geq 0$, we have the memoryless property

$$\Pr[X = n + k \mid X > k] = \Pr[X = n]$$

The following is the definition of conditional expectation.

$$\mathbb{E}[Y \mid Z = z] = \sum_{y} y \cdot \Pr[Y = y \mid Z = z],$$

a) Prove the law of total expectation below. Given any random variables X, Y, defined in the same sample space,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X|Y = y] \Pr[Y = y]$$

b) Calculate the expectation of a geometric random variable with the memoryless property and the law of total expectation.