Mathematical Foundations of Computer Science
Lecture Outline
October 1, 2024

Example. Prove that every connected graph with n vertices has at least $n-1$ edges.

Solution. We will prove the contrapositive, i.e., a graph G with $m \leq n-2$ edges is disconnected. From the result of the previous problem, we know that the number of components of G is at least

$$n - m \geq n - (n-2) = 2$$

which means that G is disconnected. This proves the claim.

One could also have proved the above claim directly by observing that a connected graph has exactly one connected component. Hence, $1 \geq n - m$. Rearranging the terms gives us $m \geq n - 1$.

Trees

A graph with no cycles is *acyclic*. A *tree* is a connected acyclic graph. A vertex of degree greater than 1 in a tree is called an *internal vertex*, otherwise it is called a *leaf*. A *forest* is an acyclic graph.

Example. Prove that every tree with at least two vertices has at least two leaves and deleting a leaf from an n-vertex tree produces a tree with $n-1$ vertices.

Solution. A connected graph with at least two vertices has an edge. In an acyclic graph, an endpoint of a maximal non-trivial path (a path that is not contained in a longer path) has no neighbors other than its only neighbor on the path. Hence, the endpoints of such a path are leaves.

Let v be a leaf of a tree T and let $T' = T - v$. A vertex of degree 1 belongs to no path connecting two vertices other than v. Hence, for any two vertices $u, w \in V(T')$, every path from u to w in T is also in T'. Hence T' is connected. Since deleting a vertex cannot create a cycle, T' is also acyclic. Thus, T' is a tree with $n-1$ vertices.

Example. For a n-vertex graph G, the following are equivalent and characterize trees with n vertices.

(1) G is a tree.
(2) G is connected and has exactly $n-1$ edges.
(3) G is minimally connected, i.e., G is connected but $G - \{e\}$ is disconnected for every edge $e \in G$.

(4) G contains no cycle but $G + \{x, y\}$ does, for any two non-adjacent vertices $x, y \in G$.
(5) Any two vertices of G are linked by a unique path in G.

Solution. **($1 \to 2$).** We can prove this by induction on n. The property is clearly true for $n = 1$ as G has 0 edges. Assume that any tree with k vertices, for some $k \geq 1$, has $k-1$ edges. We want to prove that a tree G with $k+1$ vertices has k edges. From the example we did in last class we know that G has a leaf, say v, and that $G' = G - \{v\}$ is connected. By induction hypothesis, G' has $k-1$ edges. Since $deg(v) = 1$, G has k edges.

($2 \to 3$). Note that $G - \{e\}$ has n vertices and $n-2$ edges. We know that such a graph has at least 2 connected components and hence is disconnected.

($3 \to 4$). We are assuming that removing *any* edge in G disconnects G. If G contains a cycle then removing any edge, say $\{u, v\}$, that is part of the cycle does not disconnect G as any path that uses $\{u, v\}$ can now use the alternate route from u to v on the cycle. Since G is connected there is a path from x to y in G. Let $G' = G + \{x, y\}$. G' consists of a cycle formed by the edge $\{x, y\}$ and the path from x to y in G.

($4 \to 5$). Note that since $G + \{x, y\}$ creates a cycle for for any two non-adjacent vertices in G, it must be that there must be a path between x and y in G. We will now show that there is exactly one path between any two vertices in G. We will prove this by showing that if there are two vertices that have two different paths between them then G contains a cycle. Assume that there are two paths from u to v. Beginning at u, let a be the first vertex at which the two paths separate and let b be the first vertex after a where the two paths meet. Then, there are two simple paths from a to b with no common edges. Combining these two paths gives us a cycle.