Mathematical Foundations of Computer Science
Lecture Outline
September 24, 2024

Example. The game of NIM is played as follows: Some positive number of sticks are placed on the ground. Two players take turns, removing one, two or three sticks. The player to remove the last stick loses.

A winning strategy is a rule for how many sticks to remove when there are n left. Prove that the first player has a winning strategy iff the number of sticks, n, is not $4k+1$ for any $k \in \mathbb{N}$.

Solution. We will show that if $n = 4k+1$ then player 2 has a strategy that will force a win for him, otherwise, player 1 has a strategy that will force a win for him.

Let $P(n)$ be the property that if $n = 4k+1$ for some $k \in N$ then the first player loses, and if $n = 4k, 4k+2$, or $4k+3$, the first player wins. This exhausts all possible cases for n.
Induction Hypothesis: Assume that for some $z \geq 1$, $P(j)$ is true for all j such that $1 \leq j \leq z$.
Base Case: $P(1)$ is true. The first player has no choice but to remove one stick and lose.
Induction Step: We want to prove $P(z+1)$. We consider the following four cases.
Case I: $z+1 = 4k+1$, for some k. We have already handled the base case, so we can assume that $z+1 \geq 5$. Consider what the first player might do to win: he can remove 1, 2, or 3 sticks. If he removes one stick then the remaining number of sticks $n = 4k$. By strong induction, the player who plays at this point has a winning strategy. So the player who played first loses. Similarly, if the first player removes two sticks or three sticks, the remaining number of sticks is $4(k-1)+3$ and $4(k-1)+2$ respectively. Again, the first player loses (using induction hypothesis). Thus, in this case, the first player loses regardless of what move he/she makes.
Case II: $z+1 = 4k$, or $z+1 = 4k+2$, or $z+1 = 4k+3$. If the first player removes three sticks in the first case, one stick in the second case, and two sticks in the third case then the second player sees $4(k-1)+1$ sticks in the first case and $4k+1$ sticks in the other two cases. By induction hypothesis, in each case the second player loses.

Graphs

A *graph* consists of two sets, a non-empty set, V, of vertices or nodes, and a possibly empty set, E, of 2-element subsets of V. Such is graph is denoted by $G = (V, E)$. Each element of E is called an *edge*. We say that an edge $\{u, v\} \in E$ *connects* vertices u and v. Two nodes u and v are *adjacent* if $\{u, v\} \in E$. Nodes adjacent to a vertex u are called *neighbors* of u. The number of neighbors of a vertex v is called the *degree* of v and is denoted by $deg(v)$. The value $\delta(G) = \min_{v \in V}\{deg(v)\}$ is the *minimum degree* of G, the value $\Delta(G) = \max_{v \in V}\{deg(v)\}$ is the *maximum degree* of G. An edge that connects a node to itself is called a *loop* and multiple edges between the same pair of nodes are called *parallel* edges. Graphs without loops and parallel edges are called *simple* graphs, otherwise they are called *multigraphs*. Unless specified otherwise, we will only deal with simple graphs.

Example. Prove that the sum of degrees of all nodes in a graph is twice the number of edges.

Solution. Since each edge is incident to exactly two vertices, each edge contributes two to the sum of degrees of the vertices. The claim follows.

Example. In any graph there are an even number of vertices of odd degree.

Solution. Let V_e and V_o be the set of vertices with even degree and the set of vertices with odd degree respectively in a graph $G = (V, E)$. Then,

$$\sum_{v \in V} deg(v) = \sum_{v \in V_e} deg(v) + \sum_{v \in V_o} deg(v)$$

The first term on R.H.S. is even since each vertex in V_e has an even degree. From the previous example, we know that L.H.S. of the above equation is even. Thus the second term on the R.H.S. must be even. Let $|V_o| = \ell$. We want to show that ℓ is even. Since each vertex in V_o has odd degree, we have

$$\begin{aligned}(2k_1 + 1) + (2k_2 + 1) + \cdots + (2k_\ell + 1) &\text{ is an even number}\\ 2(k_1 + k_2 + \cdots + k_\ell) + \ell &\text{ is an even number}\\ \therefore \ell &\text{ is an even number}\end{aligned}$$

This proves the claim.

A *walk* in G is a non-empty sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If the vertices in a walk are all distinct, we call it a *path* in G. Thus, a *path* in G is a sequence of distinct vertices $v_0, v_1, v_2, \ldots v_k$ such that for all i, $0 \le i < k$, $\{v_i, v_{i+1}\} \in E$. The *length* of the walk (path) is k, the number of edges in the walk (resp. path). Note that the length of the walk (path) is one less than the number of vertices in the walk (path) sequence. If $v_o = v_k$, the walk (path) is *closed*. A closed path is called a *cycle*.

The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A complete graph is a graph in which there is an edge between every pair of distinct vertices. A graph G is *connected* if there is a path in G between its every pair of vertices. A graph H is a *connected component*(“island”) of G if (a) H is a subgraph of G, (b) H is connected, and (c) H is maximal, i.e., H is not contained in any other connected subgraph of G. In short, H is a connected component of G if H is a maximal subgraph of G that is connected.

We say that H is an *induced subgraph* of a graph G if the vertex set of H is a subset of the vertex set of G, and if u and v are vertices in H, then (u, v) is an edge in H iff (u, v) is an edge in G.

Example. Prove that every graph with n vertices and m edges has at least $n - m$ connected components.

Solution. We will prove this claim by doing induction on m.
Induction Hypothesis: Assume that for some $k \geq 0$, every graph with n vertices and k edges has at least $n - k$ connected components.
Base Case: $m = 0$. A graph with n vertices and no edges has n connected components as each vertex itself is a connected component. Hence the claim is true for $m = 0$.
Induction Step: We want to prove that a graph, G, with n vertices and $k + 1$ edges has at least $n - (k + 1) = n - k - 1$ connected components. Consider a subgraph G' of G obtained by removing any arbitrary edge, say $\{u, v\}$, from G. The graph G' has n vertices and k edges. By induction hypothesis, G' has at least $n - k$ connected components. Now add $\{u, v\}$ to G' to obtain the graph G. We consider the following two cases.
Case I: u and v belong to the same connected component of G'. In this case, adding the edge $\{u, v\}$ to G' is not going to change any connected components of G'. Hence, in this case the number of connected components of G is the same as the number of connected components of G' which is at least $n - k > n - k - 1$.
CaseII: u and v belong to different connected components of G'. In this case, the two connected components containing u and v become one connected component in G. All other connected components in G' remain unchanged. Thus, G has one less connected component than G'. Hence, G has at least $n - k - 1$ connected components.