Mathematical Foundations of Computer Science
Lecture Outline
September 17, 2024

Example. Recall that for any set A, $\mathcal{P}(A)$ denotes the power set of A. Let $S = \{x_1, x_2, \dots, x_n\}$. Prove using induction that for all positive integers n, $|\mathcal{P}(S)| = 2^n$.

Solution. We will prove the claim using induction on n.
Induction Hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 1$. In other words, assume that if $S = \{x_1, x_2, \dots, x_k\}$, then $|\mathcal{P}(S)| = 2^k$.
Base Case:$n = 1$. When $S = \{x_1\}$, there are exactly two subsets of S, namely $\emptyset$ and S, itself. Thus the claim is true when $n = 1$.
Induction Step: We want to prove that the claim is true when $n = k + 1$. In other words, we want to show that if $S = \{x_1, x_2, \dots, x_k, x_{k+1}\}$, then $|\mathcal{P}(S)| = 2^{k+1}$. Let $S' = \{x_1, x_2, \dots, x_k\}$. The set of all subsets of S can be partitioned into S_1 and S_2, where $S_1 \subset \mathcal{P}(S)$ contains subsets of S that does not contain x_{k+1}, and $S_2 \subset S$ contains subsets of $\mathcal{P}(S)$ that contains x_{k+1}. Thus we have

$$|\mathcal{P}(S)| = |S_1| + |S_2| \tag{1}$$

Note that S_1 contains all subsets of $\mathcal{P}(S')$. By the induction hypothesis, we have $|S_1| = |\mathcal{P}(S')| = 2^k$. We will now compute $|S_2|$. Observe that each set in S_2 is of the form $\{x_{k+1}\} \cup X$, where X is a subset of S'. By induction hypothesis, we know that there are 2^k subsets of S' and hence $|S_2| = 2^k$. Plugging in the values for $|S_1|$ and $|S_2|$ in (1), we get

$$|\mathcal{P}(S)| = 2^k + 2^k = 2^{k+1}$$

Example Let $A_1, A_2, \dots, A_n$ be sets (where $n \geq 2$). Suppose for any two sets A_i and A_j either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Prove by induction that one of these n sets is a subset of all of them.

Solution. We will prove the claim using induction on n.
Induction Hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 2$. In other words, assume that if we have sets $A_1, A_2, \dots, A_k$, where for any two sets A_i and A_j, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$ then one of the k sets is a subset of all of the k sets.
Base Case: $n = 2$. We have two sets A_1, A_2 and we know that $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Without loss of generality assume that $A_1 \subseteq A_2$. Then A_1 is a subset of A_1 and is also a subset of A_2, so the claim holds when $n = 2$.
Induction Step: We want to prove the claim when $n = k + 1$. That is, we are given a set

$S = \{A_1, A_2, \ldots, A_{k+1}\}$ of with the property that for every pair of sets $A_i \in S$ and $A_j \in S$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. We want to show that there is a set in S that is a subset of all $k+1$ sets in S. Let $S' = S \setminus \{A_{k+1}\}$. By induction hypothesis, there is a set $A_p \in S'$ that is a subset of all sets in S'. We now consider the following two cases.
Case 1: $A_p \subseteq A_{k+1}$. Then it follows that A_p is a subset of all sets in S.
Case 2: $A_{k+1} \subseteq A_p$. Since A_p is a subset of all sets in S' and $A_{k+1} \subseteq A_p$, it follows that A_{k+1} is a subset of all sets in S.

Example. For all $n \geq 1$, prove that n lines separate the plane into $(n^2+n+2)/2$ regions. Assume that no two of these lines are parallel and no three pass through a common point.

Solution. Let $P(n)$ be the property that n lines, such that no two of them are parallel and no three of them pass through a common point, separate the plane into $(n^2+n+2)/2$ regions. We will prove the claim by induction on n.

Induction Hypothesis: Assume that $P(k)$ is true for some integer $k > 0$.
Base Case: $P(1)$ is true since one line divides the plane into 2 regions which is also given by $(1^2+1+2)/2$.
Induction Step: To prove that $P(k+1)$ is true. Consider a set S of $k+1$ lines such that no two of them are parallel and no three of them pass through a common point. Remove any line ℓ from the set S. Let S' be the resulting set of k lines. By induction hypothesis, the k lines in S' divide the plane into $(k^2+k+2)/2$ regions. Now we add the line ℓ to the set S' to obtain the set S. Line ℓ intersects exactly once with each of the k lines in S'. These intersections divide the line ℓ into $k+1$ line segments. Each of these line segments passes through a region and hence $k+1$ additional regions are created. Hence, the total number of regions formed by $k+1$ lines is given by

$$\frac{k^2+k+2}{2} + k + 1 = \frac{k^2+3k+4}{2} = \frac{k^2+2k+1+k+3}{2} = \frac{(k+1)^2+(k+1)+2}{2}$$

Thus $P(k+1)$ is correct and this completes the proof.

Example. Let n be a non-negative integer. Show that any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

Solution. (Attempt 1) Let R_n denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that R_n with one central square removed can be tiled using L-shaped pieces.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.
Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one central square removed requires 0 tiles.
Induction Step: We want to prove that $P(k+1)$ is true, i.e., region R_{k+1} with one central square removed can be tiled using L-shaped pieces.
R_{k+1} can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of

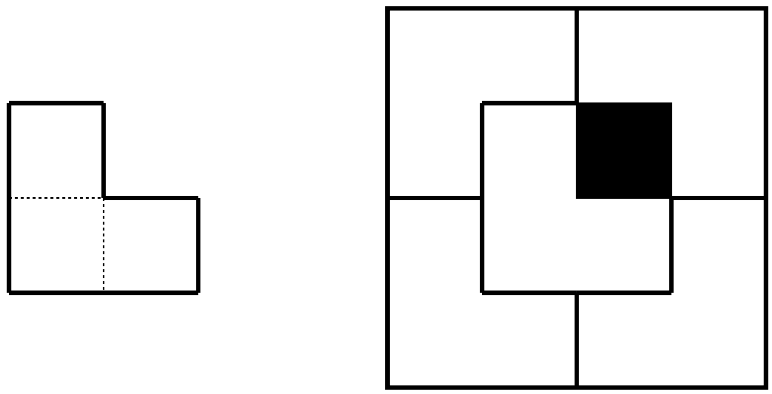

Figure 1: A L-tile and an L-tiling of a $2^2 \times 2^2$ region without a square.

R_{k+1} can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions can be covered using L-shaped tiles, R_{k+1} without a central square can also be covered using L-shaped tiles.

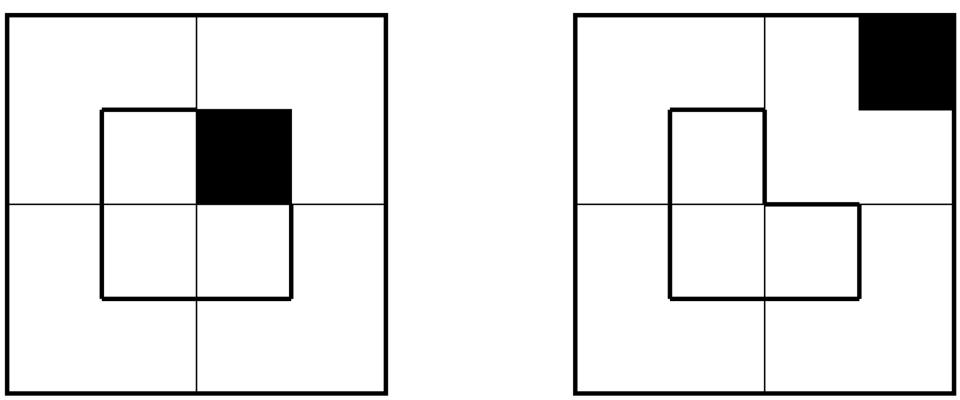

Figure 2: Illustration of the two proof attempts.

Our use of induction hypothesis is incorrect as we have assumed that region R_k without a *central* square (not a *corner* square) can be covered using L-shaped tiles.

Surprisingly, we can get around this obstacle by proving the following stronger claim.

"For all positive integers n, any R_n region with *any* one square removed can be L-tiled."

Let $P(n)$ be the property that R_n without one square can be L-tiled.
Induction Hypothesis: Assume that $P(k)$ is true for some k.
Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one square removed requires 0 tiles.
Induction Step: We want to prove that $P(k+1)$ is true, i.e., region R_{k+1} wthout one square

that is located anywhere can be L-tiled. Divide R_{k+1} into four R_k regions. One of the four R_k regions that does not have one square can be L-tiled (using induction hypothesis). Each of the other three R_k regions without the corner square that is located at the center of R_{k+1} can be L-tiled (using induction hypothesis). By using one more L-tile we can cover the three central squares of R_{k+1}.

Strong Induction.

For any property P, if $P(0)$ and $\forall n \in \mathbb{N}, P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$, then $\forall n \in \mathbb{N}, P(n)$.

Example. Prove that if n is an integer greater than 1 then either n is a prime or it can be written as a product of primes.

Solution. Let $P(n)$ be "n can be written as a product of primes".
Induction Hypothesis: Assume that $P(j)$ is true for $1 < j \leq k$.
Base Case: We want to show that $P(2)$ is true. This is clearly true as 2 is a prime.
Induction Step: We want to show that $P(k+1)$ is true.
Case I: $k+1$ is prime. In this case we are done.
Case II: $k+1$ is composite. Then,

$$k+1 = a \times b, \quad \text{for some } a \text{ and } b \text{ s.t. } 2 \leq a \leq b < k+1$$

By induction hypothesis, a is a prime or it can be written as a product of primes. The same applies to b. Since $k+1 = a \times b$, it can be written as a product of primes, namely those primes in the factorization of a and those in the factorization of b.

Example. Prove that, for any positive integer n, if $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is $n-1$.

Solution. Let $P(n)$ be the property that "If $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is $n-1$".
Induction Hypothesis: Assume that $P(j)$ is true for all j such that $1 \leq j \leq k$.
Base Case: $P(1)$ is true, since x_1 is computed using 0 multiplications.
Induction Step: We want to prove $P(k+1)$. Consider the product of $k+1$ distinct factors, $x_1, x_2, \ldots, x_{k+1}$. When parentheses are inserted in order to compute the product of factors, some multiplication must be the final one. Consider the two terms, of this final multiplication. Each one is a product of at most k factors. Suppose the first and the second term in the final multiplication contain f_k and s_k factors. Clearly, $1 \leq f_k, s_k \leq k$. Thus, by induction hypothesis, the number of multiplications to obtain the first term of the final multiplication is $f_k - 1$ and the number of multiplications to obtain the second term of the

final multiplication is $s_k - 1$. It follows that the number of multiplications to compute the product of $x_1, x_2, \ldots, x_k, x_{k+1}$ is

$$(f_k - 1) + (s_k - 1) + 1 = f_k + s_k - 1 = k + 1 - 1 = k$$