Mathematical Foundations of Computer Science
Lecture Outline
September 19, 2024

The Binomial Theorem

A binomial is a sum of two terms, such as $a+b$. The *binomial theorem* gives an expression for $(a+b)^n$ where a and b are real numbers and n is a positive integer.

Theorem. For any real numbers a and b and non-negative integer n

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. Observe that each term in the expansion of $(a+b)^n$ is of the form $a^{n-k}b^k, k = 0, 1, 2, \ldots, n$. How many terms are there of the form $a^{n-k}b^k$? This is the same number of times as there are orderings of $n-k$ a's and k b's. This is equal to $\binom{n}{k}$. Thus the coefficient of like terms of the form $a^{n-k}b^k$ is $\binom{n}{k}$. This proves the theorem.

Example. Prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Solution. Last week we proved this claim using a counting argument in which we showed that L.H.S. and R.H.S. count the number of subsets of a set of n elements. Now we will prove this using the binomial theorem as follows.

$$\begin{aligned} 2^n &= (1+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (1)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \\ &= \text{R.H.S.} \end{aligned}$$

Example. Let n be a positive integer. Then, for all x prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Solution. Using the binomial theorem we get

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Example. Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

Solution. One way to solve this problem is by substituting $x = -1$ in the previous example. When $x = -1$ the above equation becomes

$$0^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

A combinatorial proof of the claim was presented earlier.

The Pigeonhole Principle

If $k+1$ or more objects are distributed among k bins then there is at least one bin that has two or more objects. For example, the pigeon hole principle can be used to conclude that in any group of thirteen people there are at least two who are born in the same month.

Example. There are n pairs of socks. How many socks must you pick without looking to ensure that you have at least one matching pair?

Solution. The pigeonhole principle can be applied by letting n bins correspond to the n pairs of socks. If we select $n+1$ socks and put each one in the box corresponding to the pair it belongs to then there must be at least one box containing a matched pair.

The Generalized Pigeonhole Principle

If n objects are placed into k boxes, then there is at least one box containing at least $\lceil n/k \rceil$ objects.

Proof: We will prove the contrapositive. That is, we will show that if each box contains at most $\lceil n/k \rceil - 1$ objects then the total number of objects is not equal to n. Assume that each box contains at most $\lceil n/k \rceil - 1$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) < k\left(\frac{n}{k} + 1 - 1\right) = n$$

Thus we have shown that the total number of objects is less than n. This completes the proof.

Using the generalized pigeonhole principle we can conclude that among 100 people, there are at least $\lceil 100/12 \rceil = 9$ who are born in the same month.

Example. Suppose each point in the plane is colored either red or blue. Show that there always exist two points of the same color that are exactly one feet apart.

Solution. Consider an equilateral triangle with the length of each side being one feet. The three corners of the triangle are colored red or blue. By pigeonhole principle, two of these three points must have the same color.

Example. Given a sequence of n integers, show that there exists a subsequence of consecutive integers whose sum is a multiple of n.

Solution. Let $x_1, x_2, \ldots, x_n$ be the sequence of n integers. Consider the following n sums.

$$x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_n$$

If any of these n sums is divisible by n, then we are done. Otherwise, each of the n sums have a non-zero remainder when divided by n. There are at most $n - 1$ different possible remainders: $1, 2, \ldots n - 1$. Since there are n sums, by the pigeonhole principle, at least two of the n sums have the same remainder when divided by n. Let p and q, $p < q$, be integers such that for some integers c_1 and c_2,

$$x_1 + x_2 + \cdots + x_p = c_1 n + r \text{ and } x_1 + x_2 + \cdots + x_q = c_2 n + r$$

Subtracting the two sums, we get

$$x_{p+1} + \cdots + x_q = (c_2 - c_1)n$$

Hence, $x_{p+1} + \cdots + x_q$ is divisible by n.

Example. Show that in any group of six people there are either three mutual friends or three mutual strangers.

Solution. Consider one of the six people, say A. The remaining five people are either friends of A or they do not know A. By the pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of the five people are either friends of A or are unacquainted with A. In the former case, if any two of the three people are friends then these two along with A would be mutual friends, otherwise the three people would be strangers to each other. The proof for the latter case, when three or more people are unacquainted with A, proceeds in the same manner.

Example. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists consecutive days during which the chess master will have played exactly 21 games.

Solution. Let a_i, $1 \leq i \leq 77$, be the total number of games that the chess master has played during the first i days. Note that the sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence. We have

$$1 \leq a_1 < a_2 < \ldots < a_{77} \leq 11 \times 12 = 132$$

Now consider the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. We have

$$22 \leq a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \leq 153$$

Clearly, this sequence is also a strictly increasing sequence. The numbers $a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ (154 in all) belong to the set $\{1, 2, \ldots, 153\}$. By the pigeonhole principle there must be two numbers out of the 154 numbers that must be the same. Since no two numbers in $a_1, a_2, \ldots, a_{77}$ are equal and no two numbers in $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal there must exist i and j such that $a_i = a_j + 21$. Hence during the days $j+1, j+2, \ldots, i$, exactly 21 games must have been played.

Benjamin Judd suggested the following nice proof in class. For $1 \leq i \leq 77$, let g_i denote the number of games played by the chessmaster on day i. Consider the number of games played by the chessmaster during each day of the first three weeks: $g_1, g_2, \ldots, g_{21}$. By the constraints described in the question, we have

$$g_i \geq 1, i = 1, 2, \ldots, 21 \text{ and } \sum_{i=1}^{21} g_i \leq 36 \tag{1}$$

We know that in the sequence of positive integers $g_1, g_2, \ldots, g_{21}$, there must be a subsequence $S : g_l, g_{l+1}, g_{l+2}, \ldots, g_k, 1 \leq l < k \leq 21$ of consecutive integers whose sum is divisible by 21 (we proved this earlier in the lecture). Combining this with (1), we conclude that the sum of the numbers in S must be exactly 21. This means that during the days $l, l+1, l+2, \ldots, k$, the chessmaster played exactly 21 games.

Example. Prove that every sequence of $n^2 + 1$ distinct real numbers, $x_1, x_2, \ldots, x_{n^2+1}$, contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Solution. We will prove this as follows. We suppose that there is no strictly increasing subsequence of length $n+1$ and show that there must be a strictly decreasing subsequence of length $n+1$. Let $m_k, k = 1, 2, \ldots, n^2 + 1$, be the length of the longest increasing subsequence that begins with x_k. Since there is no increasing subsequence of length $n+1$, for $k = 1, 2, \ldots, n^2 + 1$, we have $1 \leq m_k \leq n$. Using the generalized pigeonhole principle, we conclude that $n+1$ of the numbers $m_1, m_2, \ldots, m_{n^2+1}$ are equal. Let

$$m_{k_1} = m_{k_2} = \cdots = m_{k_{n+1}}$$

where $1 \leq k_1 < k_2 < \cdots < k_{n+1} \leq n^2 + 1$. We will now argue that $x_{k_1} > x_{k_2} > \cdots > x_{k_{n+1}}$, which will complete the proof as we will have a decreasing subsequence of length $n+1$.

Assume for contradiction that this is not the case, which means that there is a i, $1 \leq i \leq n+1$, such that $x_{k_i} < x_{k_{i+1}}$. Then, since $k_i < k_{i+1}$, we could take a longest increasing subsequence starting with $x_{k_{i+1}}$ and put x_{k_i} in front to obtain an increasing subsequence that begins with x_{k_i}. This implies that $m_{k_i} > m_{k_{i+1}}$, which is a contradiction. Hence, for all $i = 1, 2, \ldots, n, x_{k_i} > x_{k_{i+1}}$. Thus, we have a decreasing subsequence of length $n+1$. Similarly, we can show that if there is no decreasing subsequence of length $n+1$ then there must be an increasing sequence of length $n+1$.