Mathematical Foundations of Computer Science
Lecture Outline
September 12, 2024

Example. How many permutations are there of the word MISSISSIPPI?

Solution. We want to find the number of permutations of the multiset $\{1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$. Thus, $n = 11, n_1 = 1, n_2 = 4, n_3 = 4, n_4 = 2$. Then number of permutations is given by

$$\frac{n!}{n_1!n_2!n_3!n_4!} = \frac{11!}{1!4!4!2!}$$

Example. Consider n distinct objects and k bins labeled $B_1, B_2, \ldots, B_k$. How many ways are there to distribute the objects in the bins so that bin B_i receives n_i objects and $\sum_{i=1}^{k} n_i = n$?

Solution. A partition of n objects into k labeled bins, $B_1, B_2, \ldots, B_k$ such that bin B_i gets n_i objects can be constructed in k steps. Step $i, 1 \leq i \leq k$ chooses n_i objects that go in box B_i from the remaining objects. Step $i, 1 \leq i \leq k$ can be performed in $\binom{n-n_1-n_2-\cdots-n_{i-1}}{n_i}$ ways. By the multiplication rule, the total number of ways to achieve the required partition equals

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$
$$= \frac{n!}{n_1!n_2!\cdots n_k!}$$

Another way of arriving at the solution is as follows. Let the distinct objects be numbered $1, 2, \ldots, n$. Consider the multiset $A = \{n_1 \cdot B_1, n_2 \cdot B_2, \ldots, n_k \cdot B_k\}$. The procedure of obtaining the required partition can be done in $k + 1$ steps as follows. In Step 0, we obtain a permutation P of the multiset A. In step $i, 1 \leq i \leq k$, bin B_i gets the objects corresponding to the positions of 'B_i' in P.

Step 1 can be done in $\frac{n!}{n_1!n_2!\cdots n_k!}$ ways. There is exactly one way to do each of the remaining steps. Hence, by the multiplication rule, the required answer is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Example. In how many ways can eight distinct books be divided among three students if Bill gets four books and Sharon and Marian each get two books?

Solution. Such partition can be obtained in three steps.

  Step 1. Choose 4 books for Bill out of the available 8 books.

  Step 2. Choose 2 books for Sharon out of the remaining 4 books.

  Step 3. Choose 2 books for Marian out of the remaining 2 books.

Step 1 can be performed in $\binom{8}{4}$ ways. Step 2 can be performed in $\binom{4}{2}$ ways. Step 3 can be performed in $\binom{2}{2} = 1$ way. By the multiplication rule, the total number of possible divisions is given by

$$\binom{8}{4}\binom{4}{2} = \frac{8!}{4!4!} \times \frac{4!}{2!2!} = 420.$$

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r-Combinations with Repetition Allowed.

We have seen that there are $\binom{n}{r}$ ways of choosing r *distinct* elements from a set of n distinct elements. What if we allow elements to be repeated? In other words, we want to find the number of ways there are to choose a multiset of r elements from a multiset of n distinct elements with infinite copies of each of the n elements available?

The following method was suggested in class.

A multiset of r elements can be constructed in r steps as follows. In Step i, choose one of the n elements. Since each step can be done in n ways, there are n^r multisets of r elements. Is this correct? No, this is not correct. For example, let $S = \{a, b\}$. Suppose we want to find the number of 2-combinations of S with repetition allowed. Note that the above procedure would consider the sets $\{a, b\}$ and $\{b, a\}$ as different whereas they are the same multiset and should not be counted twice. Using the above solution we get the answer as 4, but the correct answer is 3. In other words, the above procedure gives incorrect answer as it pays attention to the order of the r elements. We give the correct solution below.

Think of the n elements of the set as categories formed using $n - 1$ vertical bars (sticks). Then each multiset of size r can be represented as a string of $n-1$ vertical bars (to separate the n categories) and r crosses (to represent the r elements to be chosen). The number of crosses in each category represents the number of times the object represented by that category is chosen. Note that each multiset of size r (chosen from a multiset of n objects, with infinite copies of each object), corresponds to exactly one way to arrange the $n - 1$ sticks and r crosses and for each arrangement of $n - 1$ sticks and r crosses, there is exactly one multiset of size r. Thus the number of multisets of size r is the same as the number of permutations of the multiset $\{(n - 1) \cdot |, r \cdot \times\}$. The number of strings of $n - 1$ vertical bars and r crosses is the number of ways to choose r positions from the available $r + n - 1$ positions. The r positions chosen will contain the crosses and the remaining positions will

have the vertical bars. Thus the total number of possible ways to choose multisets of size r from a multiset of n objects with infinite copies of each object available is given by

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!}$$

Example. Consider 3 books: a computer science book, a math book, and a history book. Suppose the library has at least 6 copies of each of these books. How many ways are there to select 6 books?

Solution. The no of ways is $\binom{6+3-1}{6} = \frac{8!}{6!2!} = 28$.

Example. How many solutions are there to the equation $x_1+x_2+x_3+x_4 = 10$ if x_1, x_2, x_3, and x_4 are non-negative integers? What if each $x_i \geq 1$?

Solution. Think of x_1, x_2, x_3, and x_4 as categories in which we must place 10 $\times$'s. The number of $\times$'s in each category represents the value of the corresponding variable in the equation. The number of solutions is the number of 10 multisets of a 4-element set. This is given by

$$\binom{4+10-1}{10} = \binom{13}{10} = 286$$

If each $x_i \geq 1$, we put one $\times$ in each category to start with. Then we distribute the remaining 6 $\times$'s among the categories. Such a distribution can be represented by a string of 3 vertical bars and 6 crosses. The number of such distributions are

$$\binom{6+3}{6} = \binom{9}{6} = 84$$

Alternate Solution. We can also solve the second part of the question using PIE. For $1 \leq i \leq 4$, let X_i be the set of all possible solutions to the equation in which $x_i = 0$. Thus, the set of solutions in which at least one $x_i = 0$ is given by

$$X_1 \cup X_2 \cup X_3 \cup X_4$$

From the first part we know that the number of non-negative integer solutions to the equation is $\binom{13}{10}$. Thus the number of solutions in which each of the x_i values are at least 1 is given by

$$\binom{13}{10} - |X_1 \cup X_2 \cup X_3 \cup X_4| \tag{1}$$

By the PIE we know that

$$|X_1 \cup X_2 \cup X_3 \cup X_4| = \sum_{i=1}^{4} |X_i| - \sum_{\substack{i,j \\ i \neq j}} |X_i \cap X_j| + \sum_{\substack{i,j,k \\ i \neq j \neq k}} |X_i \cap X_j \cap X_k| - |X_1 \cap X_2 \cap X_3 \cap X_4|$$

$$= 4\binom{12}{10} - \binom{4}{2}\binom{11}{10} + 4 - 0$$

$$= 202$$

Plugging this value in equation (1), we get that the number of solutions in which each of the x_i values are greater than 0 is

$$\binom{13}{10} - 202 = 286 - 202 = 84$$

Note that this method does not scale well for large numbers, i.e., when there are many variables in the equation.

Example. What is the number of non-decreasing sequences of length 10 whose terms are taken from 1 through 25?

Solution. The procedure of constructing a non-decreasing sequence of length 10 using integers from 1 through 25 is as follows: in step 1, choose 10 numbers with repetition allowed, from $\{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot 25\}$, and in step 2, order the chosen numbers in non-decreasing order. Note that the number of ways to do Step 1 is the same as the number of permutations of the multiset $\{24 \cdot |, 10 \cdot \times\}$, since we can think of the 25 digits as 25 categories (created using 24'|') in which 10 $\times$'s are to be placed. There is exactly one way to do step 2. Thus, the total number of ways that this can be done is given by $\binom{34}{10}$.

Example. How many ways are there to choose a 5-letter strings from the 26-letter English alphabet with replacement, where strings that are anagrams are considered the same?

Solution. Let S be the set of all 5-letter strings such that if a string is in S then its anagrams are not in S. We are interested in finding $|S|$. Note that two words are anagrams of each other iff the number of occurrences of each letter in the alphabet is the same same in both words. Thus the $|S|$ is the same as the number of 5-combinations with repetitions allowed from a multiset $\{\infty \cdot a, \infty \cdot b, \infty \cdot c, \ldots, \infty \cdot z\}$. Thus $|S| = \binom{26+5-1}{5} = \binom{30}{5}$.

Combinatorial Proofs

We will prove a few identities using counting techniques. Specifically, we will use the following technique. To prove an identity we will pose a counting question. We will then answer the question in two ways, one answer will correspond to LHS and the other would correspond to the RHS. Since both answers are to the same question, the two answers must be the same.

Example. Show that $\binom{n}{r} = \binom{n}{n-r}$.

Solution. We can of course prove it algebrically. However, here is a combinatorial argument which provides more intuition. Observe that for every set of r elements that is chosen there is exactly one set of $n - r$ elements that is not chosen. Thus if a set A has k subsets of size r: $B_1, B_2, \ldots, B_k$ then each B_i can be paired up with exactly one set of size $n - r$,

namely its complement $A \setminus B_i$. Hence the number of subsets of sizaue r is same as the number of subsets of size $n - r$.

We can also prove it by answering the following counting question in two different ways.

Given a set S of n distinct elements how many r-subsets are there of the set S?

Clearly, one answer is $\binom{n}{r}$, which gives us the left hand side. Another way to solve the problem is as follows. The procedure of forming a r-subset is as follows.

Step 1: Choose the $n - r$ elements that we want to leave out.
Step 2: Include the remaining r elements in the set.

There are $\binom{n}{n-r}$ ways to do step1 and exactly one way to do step 2. Hence, by the multiplication rule, the total number of ways of chooseing r-subsets of S is $\binom{n}{n-r}$, which gives us the right hand side.

Pascal's Formula. If n and k are positive integers with $n \geq k$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. We will prove the claim by answering the following counting question in two different ways.

Given a set $X = \{x_1, x_2, \ldots, x_n\}$ of n distinct elements how many k-subsets are there of the set X?

Let S be the set of all possible k-subsets of X. Clearly, $|S| = \binom{n}{k}$, which gives us the left hand side of the claim. Another way to find $|S|$ is as follows. The set S can be partitioned into sets S_1 and S_2, where S_1 is the set of all possible k-subsets of X that contain the element x_n and S_2 is the set of all possible k-subsets of X that do not contain the element x_n. In any k-subset of X that is in S_1, the other $k - 1$ elements (since x_n is already in the subset) come from $X \setminus \{x_n\}$. Since there are $\binom{n-1}{k-1}$ ways of choosing these subsets, $|S_1| = \binom{n-1}{k-1}$. The k elements of any set in S_2 must be chosen from $X \setminus \{x_n\}$. There are $\binom{n-1}{k}$ ways of doing this. Since S_1 and S_2 partition the set S, we have

$$|S| = |S_1| + |S_2| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This gives us the right hand side of the claim.

Example. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Solution. We pose the following counting question.

Given a set S of n distinct elements how many subsets are there of the set S?

From earlier lectures, we know that the answer is 2^n. This gives us the RHS.

Another way to compute the answer to the question is as follows. The power set $\mathcal{P}(S)$ containing all possible subsets can be partitioned into $S_0, S_1, \ldots, S_n$, where S_i, $0 \leq i \leq n$, is the set of all subsets of S that have cardinality i. Thus

$$
\begin{aligned}
|\mathcal{P}(S)| &= |S_0| + |S_1| + \ldots + |S_n| \\
&= \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} \\
&= \sum_{k=0}^{n} \binom{n}{k} = \text{LHS}
\end{aligned}
$$

This proves the claim.

Example. Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Solution. We pose the following counting question.

How many ways are there to choose two numbers from $S = \{0, 1, 2, \ldots, n\}$?

By definition, there are $\binom{n+1}{2} = \frac{n(n+1)}{2}$ distinct pairs of S. This gives us the RHS.

We can also compute the answer as follows. Let P be the set of all pairs of S. P can be partitioned into $S_1, S_2, \ldots, S_n$, where S_i, $1 \leq i \leq n$, is the set of pairs in which i is the bigger element in the pair. Clearly,

$$
\begin{aligned}
|P| &= |S_1| + |S_2| + \ldots + |S_n| \\
&= 1 + 2 + \ldots + n \\
&= \sum_{k=1}^{n} k = \text{LHS}
\end{aligned}
$$

This proves the claim.

Example. Give a combinatorial proof to show that

$$
\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}
$$

Solution. We pose the following counting question.

There are n men and m women, where $n \geq r$ and $m \geq r$. How many ways are there to form a committee of r people from this group of people?

By definition, there are $\binom{n+m}{r}$ distinct committees of r people. This gives us the RHS.

The set S of all possible committees of r people can be partitioned into subsets $S_0, S_1, S_2, \ldots, S_r$, where S_k is the set of committees in which there are exactly k men and the rest $r - k$ are women. Note that $|S_k| = \binom{n}{k}\binom{m}{r-k}$. Thus we have

$$|S| = \sum_{k=0}^{r} |S_k|$$
$$= \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

which gives us the left hand side of the expression.

The Binomial Theorem

A binomial is a sum of two terms, such as $a + b$. The *binomial theorem* gives an expression for $(a + b)^n$ where a and b are real numbers and n is a positive integer.

Theorem. For any real numbers a and b and non-negative integer n

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. Observe that each term in the expansion of $(a + b)^n$ is of the form $a^{n-k}b^k$, $k = 0, 1, 2, \ldots, n$. How many terms are there of the form $a^{n-k}b^k$? This is the same number of times as there are orderings of $n - k$ a's and k b's. This is equal to $\binom{n}{k}$. Thus the coefficient of like terms of the form $a^{n-k}b^k$ is $\binom{n}{k}$. This proves the theorem.