Mathematical Foundations of Computer Science
Lecture Outline
August 29, 2024

Counting

Counting is a part of combinatorics, an area of mathematics which is concerned with the arrangement of objects of a set into patterns that satisfy certain constraints. We will mainly be interested in the number of ways of obtaining an arrangement, if it exists.

Before we delve into the subject, let's take a small detour and understand what a *set* is. Below are some relevant definitions.

- A *set* is an *unordered* collection of distinct objects. The objects of a set are sometimes referred to as its elements or members. If a set is finite and not too large it can be described by listing out all its elements, e.g., $\{a, e, i, o, u\}$ is the set of vowels in the English alphabet. Note that the order in which the elements are listed is not important. Hence, $\{a, e, i, o, u\}$ is the same set as $\{i, a, o, u, e\}$. If V denotes the set of vowels then we say that e belongs to the set V, denoted by $e \in V$ or $e \in \{a, e, i, o, u\}$.

- Two sets are *equal* if and only if they have the same elements.

- The *cardinality* of S, denoted by $|S|$, is the number of distinct elements in S.

- A set A is said to be a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to denote that A is a subset of the set B, e.g., $\{a, u\} \subseteq \{a, e, i, o, u\}$. Note that for any set S, the empty set $\{\} = \emptyset \subseteq S$ and $S \subseteq S$. If $A \subseteq B$ and $A \neq B$ the we say that A is a *proper subset* of B; we denote this by $A \subset B$. In other words, A is a proper subset of B if $A \subseteq B$ and there is an element in B that does not belong to A.

- A *power set* of a set S, denoted by $P(S)$, is a set of all possible subsets of S. For example, if $S = \{1, 2, 3\}$ then $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. In this example $|P(S)| = 8$.

- Some of the commonly used sets in discrete mathematics are: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathbb{Q} = \{p/q \,|\, p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ ,and } q \neq 0\}$, and $\mathbb{R}$ is the set of real numbers.

- Another way to describe a set is by explicitly stating the properties that all members of the set must have. For instance, the set of all positive even integers less than 100 can be written as $\{x \,|\, x \text{ is a positive even integer less than } 100\}$ or $\{x \in \mathbb{Z}^+ \,|\, x < 100 \text{ and } x = 2k, \text{ for some integer } k\}$. Similarly, the set $\{2, 4, \ldots, 12\}$ can be written as $\{2n \,|\, 1 \leq n \leq 6 \text{ and } n \in \mathbb{N}\}$ or $\{n + 1 \,|\, n \in \{1, 3, 5, 7, 11\}\}$.

Understanding the above terminology related to sets is enough to get us started on counting.

Theorem. If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example. How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

Solution. The first number in the range that divisible by 5 is 100 (5×20) and the last one that is divisible by 5 is 995 (5×199). Using the above theorem, there are $199 - 20 + 1 = 180$ numbers from 100 to 999 that are divisible by 5.

Tree Diagram. A tree diagram is a very useful tool for systematically keeping track of all possible outcomes of a combinatorial process. We will also use this tool when we study probability.

Example. Teams A and B are to play each other in a best-of-three match, i.e., they play each other until one team wins two games in a row or a total of three games are played. What is the number of possible outcomes of the match? What does the possibility tree look like if they play three games regardless of who wins the first two?

Solution. The possibility trees for the two cases are shown in Figure 1. From the tree diagram it is clear that there are 6 outcomes in the first case and 8 in the second case.
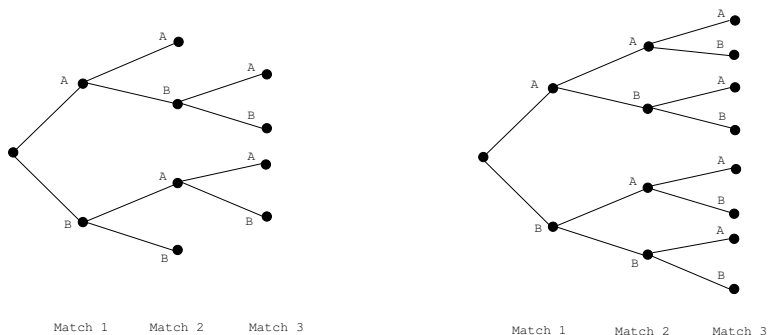

Figure 1: Tree diagrams.

Multiplication Rule. If a procedure can be broken down into k steps and

> the first step can be performed in n_1 ways,
> the second step can be performed in n_2 ways, regardless of how the first step was performed,
> $\vdots$
> the kth step can be performed in n_k ways, regardless of how the preceding steps were performed, then

the entire procedure can be performed in $n_1 \cdot n_2 \cdots n_k$ ways.

To apply the multiplication rule think of objects that you are trying to count as the output of a multi-step operation. The possible ways to perform a step may depend on how the preceding steps were performed, but the number of ways to perform each step must be constant regardless of the action taken in prior steps.

Example. An ordered pair (a, b) consists of two things, a and b. We say that a is the first member of the pair and b is the second member of the pair. If M is an m-element set and N is an n-element set, how many ordered pairs are there whose first member belongs to M and whose second member belongs to N?

Solution. An ordered pair can be formed using the following two steps.

    Step 1. Choose the first member of the pair from the set M.

    Step 2. Choose the second member of the pair from the set N.

Step 1 can be done in m ways and Step 2 can be done in n ways. From the multiplication rule it follows that the number of ordered pairs is mn.

Example. A local deli that serves sandwiches offers a choice of three kinds of bread and five kinds of filling. How many different kinds of sandwiches are available?

Solution. A sandwich can be made using the following two steps.

    Step 1. Choose the bread.

    Step 2. Choose the filling.

Step 1 can be done in 3 ways and Step 2 can be done in 5 ways. From the multiplication rule it follows that the number of available sandwich offerings is 15.

Example. The chairs of an auditorium are to be labeled with a upper-case letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution. A chair can be labeled using the following two steps.

    Step 1. Choose the upper-case letter.

    Step 2. Choose the number.

Step 1 can be done in 26 ways and Step 2 can be done in 100 ways. From the multiplication rule it follows that the number of possible labelings is 2600.

Example. A typical PIN is a sequence of any four symbols chosen from 26 letters in the alphabet and the 10 digits, with repetition allowed. How many different PINS are possible? What happens if repetition is not allowed?

Solution.   A PIN can be formed using the following steps.

   Step 1. Choose the alphanumeric for the first position.

   Step 2. Choose the alphanumeric for the second position.

   Step 3. Choose the alphanumeric for the third position.

   Step 4. Choose the alphanumeric for the fourth position.

When repetition is allowed, each step can be done in 36 ways and hence the number of possible PINS is 36^4. When repetition is not allowed, the number of ways of doing Step 1 is 36, the number of ways of doing Step 2 is 35, the number of ways of doing Step 3 is 34, and the number of ways of doing Step 4 is 33. By multiplication rule, the number of PINs in this case is $36 \times 35 \times 34 \times 33$.

Example.   Three officers - a president, a treasurer, and a secretary - are to be chosen from among four people: Ann, Bob, Clyde, and Dan. Suppose that for various reasons, Ann cannot be the president and either Clyde or Dan must be the secretary. In how many ways can the officers be chosen?

Solution.   Attempt 1. A set of three officers can be formed as follows.

   Step 1. Choose the president.

   Step 2. Choose the treasurer.

   Step 3. Choose the secretary.

There are 3 ways to do Step 1. There are 3 ways of doing Step 2 (all except the person chosen in Step 1), and 2 ways of doing Step 3 (Clyde or Dan). By multiplication rule, the number of different ways of choosing the officers is $3 \times 3 \times 2 = 18$.
    The above solution is incorrect because the number of ways of doing Step 3 depends upon the outcome of Steps 1 and 2 and hence the multiplication rule cannot be applied. It is easy to see this from the tree diagram in Figure 2.
    Attempt 2. A set of three officers can be formed as follows.

   Step 1. Choose the secretary.

   Step 2. Choose the president.

   Step 3. Choose the treasurer.

Step 1 can be done in 2 ways (Clyde or Dan). Step 2 can be done in 2 ways (Ann cannot be the president and the person chosen in Step 1 cannot be the president). Step 3 can be done in 2 ways (either of the two remaining people can be the treasurer). By multiplication rule, the numberof ways in which the officers can be chosen is $2 \times 2 \times 2 = 8$.

From the previous example we learn that there may not be a fixed order in which the operations have to be performed, and by changing the order a problem may be more readily solved by the multiplication rule. A rule of thumb to keep in mind is to *make the most restrictive choice first*.
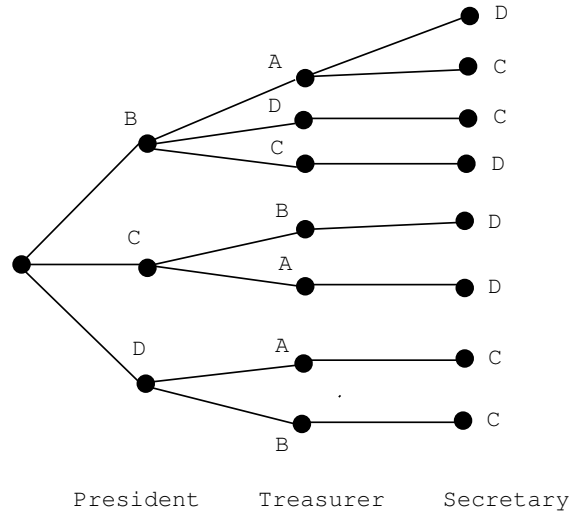
Figure 2: Tree diagram. In the tree A, B, C, stand for Ann, Bob, and Clyde respectively.

Example. Recall that the power set $P(S)$ of a set S is the set of all possible subsets of S. If $S = \{x_1, x_2, \ldots, x_n\}$, what is $P(S)$?

Solution. A subset of S can be constructed in n steps such that in step $i, 1 \leq i \leq n$, we decide whether to choose x_i or not. Each step can be performed in 2 ways regardless of the decisions made in the previous steps. By using the multiplication rule, the total number of subsets of S equals 2^n.

Example. How many odd numbers between 1000 and 9999 have distinct digits?

Solution.
Attempt 1: An odd number from 1000 through 9999 can be constructed in four steps as follows.

   Step 1. Choose the first digit.
   Step 2. Choose the second digit.
   Step 3. Choose the third digit.
   Step 4. Choose the fourth digit.

Observe that the number of ways of performing Step 4 depends upon the choices made in the earlier steps. For example, if the choices made in the first three steps are 1, 3, and 5, then Step 4 can be performed in two ways. However, if the choices made in the first three steps are 2, 4, and 6 then Step 4 can be performed in five ways. Hence, we cannot apply multiplication rule to solve the problem in the above manner.

Attempt 2: An odd number from 1000 through 9999 can be constructed in four steps as follows.

Step 1. Choose the fourth digit.

Step 2. Choose the third digit.

Step 3. Choose the second digit.

Step 4. Choose the first digit.

Note that the number of ways of performing Step 4 depends upon whether a zero was chosen in the earlier steps. If a zero was chosen in either Step 2 or Step 3 then the number of ways of performing Step 4 is 7, otherwise it is 6. Hence, multiplication rule cannot be applied to solve the problem in the above manner.

Attempt 3. An odd number from 1000 through 9999 can be constructed in four steps as follows.

Step 1. Choose the fourth digit.

Step 2. Choose the first digit.

Step 3. Choose the second digit.

Step 4. Choose the third digit.

There are 5 ways to perform Step 1, 8 ways to perform Step 2, 8 ways to perform Step 3, and 7 ways to perform Step 4. Note that the number of ways of doing each step is independent of the choices made in the earlier steps. By the multiplication rule, the number of odd numbers from 1000 through 9999 equals $5 \times 8 \times 8 \times 7 = 2240$.

Q. How many even numbers between 1000 and 9999 have distinct digits? Note that the solution to the above problem does not work for this one.

Permutations.

A permutation of a set of distinct objects is an ordering of the objects in a row. For example, the set of elements x, y, and z has six permutations: $xyz, xzy, yxz, yzx, zxy, zyx$.

In general, how many permutations are possible if we have a set of n distinct objects?

A permutation can be obtained in a sequence of n steps such that in step $i, 1 \le i \le n$, we choose the ith element in the ordering. Note that step i can be performed in i ways regardless of the choices made in the first $i - 1$ steps. By multiplication rule, the number of permutations is
$$n \times n - 1 \times n - 2 \times \cdots \times 2 \times 1 = n!.$$

Example. Consider the set of letters $\{a, b, c, d, e, f, g, h\}$. (a) How many possible permutations are there of these letters? (b) How many permutations of these letters contain the substring abc?

Solution.
(a) There are 8 distinct elements and hence 8! permutations.
(b) We consider the string *abc* as one unit and that along with the remaining elements amounts to 6 distinct elements. Hence there are 6! possible permutations.

The following question was raised in class. Can we solve part (b) from first principles? We can do it as follows.
A permutation of letters consisting of substring *abc* can be constructed in eight steps as follows. In Steps 1, 2, and 3, choose the positions for a, b, and c, respectively. In Step i, $4 \leq i \leq 8$, choose position for the ith element in the set. Step 1 can be performed in 6 ways as a can be placed only in the first six positions. Choosing a position for a also decides positions for b and c. Hence, Steps 2 and 3 can be performed in exactly 1 way. Step i, $4 \leq i \leq 8$ can be performed in $8 - (i-1)$ ways regardless of the choices made in the earlier steps. By the multiplication rule, the number of required permutations is given by

$$6 \times 1 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$

Permutations of Selected Elements.

We looked at permutations of n elements out of the available n elements. Now we will consider permutations of r elements out of the available n elements. Such an arrangement is called an r-permutation. For example, ab, ba, ac, ca, bc, cb are all 2-permutations of the set $\{a, b, c\}$.

Let $P(n, r)$ denote the number of r-permutations of a set of n elements. What is the value of $P(n, r)$?

Forming an r-permutation of a set of n elements can be thought of as an r-step process such that in step $i, 1 \leq i \leq r$, we choose the ith element of the ordering. There are $n - (i-1) = n - i + 1$ ways of performing step i. By the multiplication rule, the number of r-permutations equals

$$
\begin{aligned}
P(n, r) &= n \times n - 1 \times n - 2 \times \cdots \times n - (r-1) \\
&= n \times n - 1 \times n - 2 \times \cdots \times n - r + 1 \\
&= \frac{n \times (n-1) \times \cdots \times (n-r+1) \times (n-r) \times \cdots \times 1}{n - r \times (n-r-1) \times (n-r-2) \times \cdots \times 1} \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

Example. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different contestants?

Solution. Selecting the winners can be done in 3 steps with each step $i, 1 \leq i \leq 3$ choosing the winner in the ith place. Step i can be performed in $100 - (i-1)$ ways. By

multiplication rule, the total number of possible ways in which the prizes can be given is $100 \times 99 \times 98 = 970200$. Note that this is same as $P(100, 3)$.