Mathematical Foundations of Computer Science
Lecture Outline
December 3, 2024

Corollary Let $G = (V, E)$ be a planar graph with at least two edges that does not contain a cycle with 3 vertices. Then $|E| \le 2|V| - 4$.

Proof. We can repeat the same analysis as above, only we can obtain a tighter bound on the total degree. If G' is acyclic then $f = 1$ and the degree of the one face is at least $4 \ge 4f$. Otherwise, there is a cycle on each face's border, and because the graph has no cycle with 3 vertices, each cycle must have at least 4 edges, so the total degree is at least $4f$.

(because we assumed that G had at least two edges). Otherwise, there is a cycle on each face's border, and cycles have at least 3 edges, so the total degree is at least $3f$.

Thus,

$$2|E'| \ge 4f = 4 \cdot (2 + |E'| - |V|) = 8 + 4|E'| - 4|V|$$

Re-arranging, $2|E'| \le 4|V| - 8$, so $|E'| \le 2|V| - 4$, so $|E| \le 2|V| - 4$.

Characterizing Planar Graphs

Example. Let G be a planar graph with minimum degree δ. Then $\delta \le 5$.

Proof. Assume for the purpose of contradiction that $G = (V, E)$ is a planar graph with minimum degree $\delta > 5$. Because $\delta > 5$ and δ is an integer, $\delta \ge 6$, so

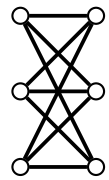
$$2|E| = \sum_{v \in V} \deg(v) \ge \sum_{v \in V} 6 = 6 \cdot |V|$$

so $2|E| \ge 6 \cdot |V|$. Because G is planar, $|E| \le 3|V| - 6$, so $2 \cdot (3|V| - 6) \ge 6|V|$. But $2 \cdot (3|V| - 6) = 6|V| - 12$, so $6|V| - 12 \ge 6|V| \implies -12 \ge 0$, a contradiction.

Example. The graph K_5 (the complete graph on 5 vertices) is not planar.

Proof. If K_5 were planar, then $|E| \le 3|V| - 6$, But in K_5, $|V| = 5$ and $|E| = \binom{5}{2} = 10$. $10 > 3 \cdot 5 - 6 = 9$, so $|E| > 3|V| - 6$, so K_5 is not planar.

Example. The graph $K_{3,3}$ (the complete bipartite graph where $|X| = |Y| = 3$, pictured below) is not planar.

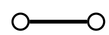


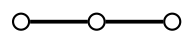
Proof. Note that $K_{3,3}$ is bipartite (the left 3 and right 3 vertices in the above drawing are independent sets). We proved that $K_{3,3}$ is bipartite, so it does not have any odd-length cycles; in particular, it does not contain a cycle on three vertices. If $K_{3,3}$ were planar, by the second corollary above $|E| \leq 2|V| - 4$. But $|E| = 9$ and $|V| = 6$, and $9 > 2 \cdot 6 - 4 = 8$, so $K_{3,3}$ cannot be planar.

To characterize *all* planar graphs, we will need one additional definition:

Definition. A *subdivision* of a graph G is formed from G by replacing edges with paths, adding additional vertices as needed.

For example, the graph on the right is a subdivision of the graph on the left:





Theorem (Kuratowski). A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

You can show the easier direction of the claim using the fact that K_5 and $K_{3,3}$ are not planar by observing that subgraphs of planar graphs must be planar and if a graph is not planar then its subdivisions must also be nonplanar.

The other direction is more difficult, and requires much more advanced graph-theoretic tools than we have developed so far.

Coloring Planar Graphs

In the 70s, Appel and Haken proved the following remarkable result:

Theorem (Four color). If G is planar, then $\chi(G) \leq 4$.

The proof is not at all intuitive, and was one of the first proofs to take advantage of computers to check an exhaustive list of cases.

We can, however, prove a weaker version of the theorem:

Example. If G is planar, then $\chi(G) \leq 6$.

Proof. We will prove the claim by induction on the number of vertices.

<u>Induction Hypothesis:</u> Assume that $\chi(G) \leq 6$ for all graphs G on k vertices, for some $k \in \mathbb{Z}^+$.

<u>Base Case:</u> When $n \leq 6$, the graph can be 6-colored by assigning each vertex in the graph a different color.

<u>Induction Step:</u> Let G be a planar graph with $k+1$ vertices. Above, we proved that $\delta(G) \leq 5$, so there must be some vertex $v \in G$ such that $\deg(v) \leq 5$. $G - v$ is a planar graph on k vertices, so by the Induction Hypothesis it can be colored using 6 colors. Using this coloring, we obtain a 6-coloring of G by using this coloring, and then coloring v a different color from all of its neighbors. Because $\deg(v) \leq 5$, there is always at least one color that is not used by any of v's neighbors, so we can always do this. Therefore, G is 6-colorable, so $\chi(G) \leq 6$.

A fundamental question in graph coloring is: what is the relation between $\chi(G)$ and the size of the largest clique? We now show that simply bounding the size of the largest clique does not allow us to bound $\chi(G)$.

Example. For any $k \geq 1$, there exist triangle-free graphs (size of the largest clique is at most 2) with chromatic number greater than k.

Solution. Let $G = (n, p)$ be a n-vertex graph, in which an edge between any two vertices is included with a probability of p.
Note that if $\chi(G) = k$ then there must be an independent set in G of size $\lceil n/k \rceil$. Thus, to show that $\chi(G) \geq k$, it suffices to show that the largest independent set in G is at most $\lceil n/k \rceil$. We will show that with a high probability, for a suitable value of p, G does not have an independent set of size $\lceil n/2k \rceil$.
Let I be the random variable denoting the number of independent sets of size $\lceil n/2k \rceil$ in G. For any set S consisting of $\lceil n/2k \rceil$ vertices, let I_S be an indicator random variable that is 1, iff S is an independent set. Thus we have

$$
\begin{aligned}
\mathbf{E}[I_S] &= \Pr[I_S = 1] \\
&= (1-p)^{\binom{\lceil n/2k \rceil}{2}} \\
&\leq (1-p)^{\binom{n/2k}{2}} \\
&= (1-p)^{\frac{(n/2k)(n/2k-1)}{2}} \\
&= e^{-p(\frac{n^2}{8k^2} - \frac{n}{4k})} (\text{using } 1 + x \leq e^x, \text{ for all } x) \\
&\leq e^{-p(\frac{n^2}{16k^2})} \quad (\text{for } n \geq 4k) \\
&< 2^{-\frac{n^{1+\epsilon}}{16k^2}} \qquad (1)
\end{aligned}
$$

The expected value of I can now be calculated as follows.

$$
\begin{aligned}
I &= \sum_S I_S \\
\mathbf{E}[I] &= \sum_S \mathbf{E}[I_S] \\
&< \sum_S 2^{-\frac{n^{1+\epsilon}}{16k^2}} \qquad\qquad \text{(using (1))} \\
&= \binom{n}{\lceil n/2k \rceil} 2^{-\frac{n^{1+\epsilon}}{16k^2}} \\
&< 2^n \times 2^{-\frac{n^{1+\epsilon}}{16k^2}} \\
&= 2^{n(1-\frac{n^\epsilon}{16k^2})}
\end{aligned}
$$

We want $\mathbf{E}[I] \leq 1/2$. For this to happen, it suffices that

$$
\begin{aligned}
n(1-\frac{n^\epsilon}{16k^2}) \quad & \leq -1, \text{which holds if} \\
n - \frac{n^{1+\epsilon}}{16k^2} \quad & \leq -1, \text{which holds if} \\
n+1 \quad & \leq \tfrac{n^{1+\epsilon}}{16k^2}, \text{which holds if} \\
2n \quad & \leq \tfrac{n^{1+\epsilon}}{16k^2}, \text{which holds if} \\
n \quad & \leq \tfrac{n^{1+\epsilon}}{32k^2}, \text{which holds if} \\
n^\epsilon \quad & \geq 32k^2, \text{which holds if} \\
n \quad & \quad\quad \geq (32k^2)^{1/\epsilon}
\end{aligned} \tag{2}
$$

Thus, we have that for all $n \geq (32k^2)^{\frac{1}{\epsilon}}$, $\mathbf{E}[I] < 1/2$. By Markov's inequality, we have

$$
\Pr[I \geq 1] \leq \mathbf{E}[I] < \frac{1}{2}
$$

Let T be the random variable denoting the number of triangles. Fix a set of 3 vertices; the probability that they form a triangle is p^3. Summing this over all 3-subsets, we get

$$
\begin{aligned}
\mathbf{E}[T] &= \binom{n}{3} p^3 \\
&< \frac{n^3}{3!}(n^{\epsilon-1})^3 \\
&= \frac{n^{3\epsilon}}{6}
\end{aligned}
$$

Using Markov's inequality, we have

$$
\Pr[T \geq n/2] \leq \frac{\mathbf{E}[T]}{n/2} < \frac{n^{3\epsilon}/6}{n/2} = \frac{1}{3n^{1-3\epsilon}}
$$

Setting the last expression to be at most 1/3, we have

$$\begin{aligned} \frac{1}{3n^{1-3\epsilon}} &\le 1/3 \\ \epsilon &\le 1/3 \end{aligned}$$

By plugging $\epsilon = 1/3$ in (2), we get $n \geq 2^{15}k^6$. Thus, we have that for all $n \geq 2^{15}k^6$, we have $\Pr[I \geq 1] + \Pr[T \geq n/2] < 1$. This means that there exists a graph G for which $I = 0$ and $T < n/2$. We now alter this graph G by deleting one vertex from each triangle in G. Let G' be the resulting triangle-free graph. We remove less than $n/2$ vertices from G, thus G' has at least $n/2$ vertices. Since G does not have an independent set of size $\lceil n/2k \rceil$, G' does not have an independent set of size $\lceil n/2k \rceil \leq \lceil |G'|/k \rceil$. Thus $\chi(G') > k$.

The *girth* of a graph G, $g(G)$, is the length of the smallest cycle in G. In triangle-free graphs, $g(G) > 3$. In 1954 B. Descartes constructively showed that triangle-free graphs can have high chromatic number, but this construction was complicated and contained many short cycles. In 1959, Paul Erdős used the probabilistic method to prove the existence of graphs with arbitrarily high girth and chromatic number.

Example (Erdős 1959) For every $g, k > 0$, there exists a graph G with $\chi(G) \geq k$ and $g(G) \geq g$.