Mathematical Foundations of Computer Science
Lecture Outline
November 25, 2024

Recall that a tournament is a directed graph with exactly one directed edge between any pair of vertices. A tournament $G = (V, E)$ is called k-dominated if for every set of k vertices $v_1, v_2, \ldots, v_k$, there exists another vertex $u \in V$ such that $(u, v_i) \in E$, for $i = 1, 2, \ldots, k$.

Example. Prove that for any positive integer k, if n is large enough then there is a k-dominated tournament on n vertices. For sufficiently large values of k, $n = k^2 2^k$ suffices.

Solution. Note that for $k = 1$, a tournament on three vertices that is a directed 3-cycle is 1-dominated. We assume $k \geq 2$ for the rest of the proof. Construct a random tournament G in which an edge between any two vertices u and v is directed towards u with probability $\frac{1}{2}$ and towards v with probability $\frac{1}{2}$. The bad event for our random process is that G is not k-dominated. We will calculate the probability of this bad event as follows. Let S be a fixed set of k vertices in G. The probability that a vertex u outside of S does not dominate set S is given by $1 - (1/2)^k$. Thus the probability that S is not dominated by any of the $n - k$ vertices outside of S is given by $(1 - 1/2^k)^{n-k}$. Since there are $\binom{n}{k}$ possibilities for set S, the probability of some set of k vertices in G not being dominated is at most

$$\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} \tag{1}$$

If the above expression is less than 1, it means that the probability of the random tournament G being k-dominated is strictly larger than 0, which means that such a tournament exists. We will now show that if $n/\ln n > k2^k$ then the expression (1) is less than 1.

$$\begin{aligned}
\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} &\leq \frac{n^k}{k!} \cdot e^{\frac{k-n}{2^k}} && (\text{using } 1 + x \leq e^x, \forall x \in \mathbb{R}, \ \ \texttt{Note:} \ \ -(n-k) = k - n) \\
&= \frac{e^{k \ln n}}{k!} \cdot e^{\frac{k}{2^k} - \frac{n}{2^k}} && (\text{since } n = e^{\ln n}) \\
&= \frac{e^{\frac{k}{2^k}}}{k!} \cdot e^{k \ln n - \frac{n}{2^k}} \\
&\leq \frac{e^{\frac{k}{2k}}}{k!} \cdot e^{k \ln n - \frac{n}{2^k}} && (\text{since } 2^k \geq 2k \text{ for positive } k) \\
&\leq \frac{\sqrt{e}}{k!} \cdot 1 (\text{since } n/\ln n > k2^k, \text{ we have } n/2^k > k \ln n, \text{ hence } 0 < e^{k \ln n - \frac{n}{2^k}} < 1) \\
&< \frac{2}{k!} < 1, && \text{for all } k \geq 2
\end{aligned}$$

Note that for large values of k, $n > k^2 2^k$ satisfies the inequality $n/\ln n > k2^k$. This is because when $n = k^2 2^k$, we have

$$\frac{n}{\ln n} = \frac{k^2 2^k}{\ln(k^2 2^k)}$$

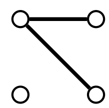
Note that for the last term to be larger than $k2^k$, it must be that

$$\ln(k^2 2^k) < k \Rightarrow k^2 2^k < e^k \Rightarrow k^2 < \left(\frac{e}{2}\right)^k$$

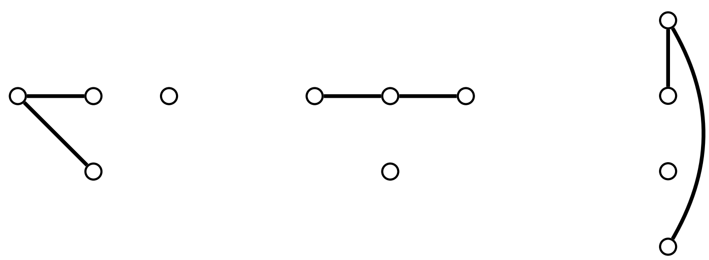
which is true for sufficiently large values of k.

Planar Graphs

In previous lectures, we have sometimes thought of graphs as looking like this:



In reality, however, a graph $G = (V, E)$ is nothing more than a finite non-empty set of vertices V and an associated set of edges E; while these drawings help capture some of the properties of a graph, they are not *equal* to the original graph. Indeed, there are often many ways to draw the same graph:



All of these pictures represent the same graph — they have the same vertex set and edge set — and yet in some sense each are different. In today's lecture we are going to study *embeddings*, which are a mathematical way to describe representations of graphs, like the ones shown above.

Graph Embeddings

With this motivation in mind, we can now define what an embedding is.

Definition. An *embedding* of a graph $G = (V, E)$ is a collection of points and curves in a plane satisfying the following criteria:

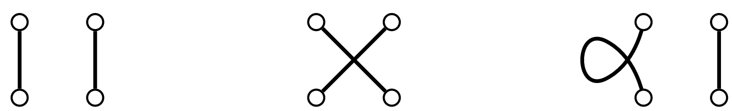
- Each vertex $v \in V$ is assigned exactly one point $p(v)$ in the plane. If we have two vertices $u, v \in V$ where $u \neq v$, $p(u) \neq p(v)$.
- Each edge $\{u, v\} \in E$ is assigned to a curve in the plane with endpoints $p(u)$ and $p(v)$, and the curve cannot pass through any other points $p(w)$ for another vertex $w \in V$.

The above pictures are depictions of multiple different embeddings of the same graph in the plane.

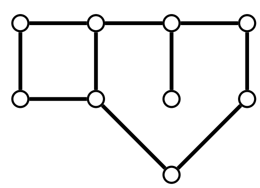
In graph theory, it turns out that there is an interesting class of embeddings that have a number of special properties that we are going to explore today:

Definition. An embedding is *crossing-free* if all curves are *simple* (do not cross themselves) and the curves associated with any two distinct edges e and e' do not intersect except at their endpoints.

Here are three embeddings of the same graph in the plane, one of which is crossing-free and two of which are not:



Definition. Given a crossing-free embedding of a graph, a *face* is a region of the plane that is cut off from the other faces by edges. For example, the following graph has 3 faces:



where two of the faces are the two enclosed regions and the third face is the rest of the plane that is not part of the other two faces. We call this face the *unbounded face*, and all of the other faces are *bounded*.

Warning: Some of the concepts touched upon today, particularly those related to curves, simple curves, and curve crossings in the plane are not going to be spelled out in full detail. To do so would require much of a course on topology. Instead, we will rely on intuitive notions of what curves look like in a plane, and omit the detailed topological reasoning.

Planar Graphs

With these definitions in mind, we can get to the main focus of today's lecture.

Definition. A *planar graph* $G = (V, E)$ is a graph that has at least one crossing-free embedding in the plane.

Theorem (Euler's Formula). Let G be a connected planar graph with n vertices and m edges. For any crossing-free embedding of G,

$$n - m + f = 2$$

where f is the number of faces.

Proof. We will prove the claim using induction on m. Because in the claim G is connected, we know that $m \geq n-1$, so this will be our base case.
Induction Hypothesis: Assume that the property holds for any graph G with k edges, for some integer $k \geq n-1$.
Base Case: When $m = n-1$, G is a tree. Fix a particular crossing-free embedding for G, and let f be the number of faces. Observe that, in order for a graph to have a bounded face, the edges forming the outer border of this face must form a cycle (otherwise they would not fully enclose the face). This means that, because G is acyclic, there cannot be any bounded faces in the embedding, so $f = 1$, which means that

$$n - m + f = n - (n-1) + 1 = 2$$

Induction Step: Let G be a connected planar graph with n vertices and $k+1$ edges, and fix a particular crossing-free embedding for G. Let f be the number of faces in the embedding. Because G is connected and has $k+1 \geq n$ edges, it is not a tree, so it must have a cycle C. Let e be any edge in C, and consider what happens when we remove the edge e from G to create a new graph G'. We can turn our crossing-free embedding for G into a crossing-free embedding for G' by removing the line segment corresponding to e. Because G' has k edges, we can apply the induction hypothesis, which states that $n - k + f' = 2$, where f' is the number of faces in G''s embedding.

When we add e back to G', we split a face in the embedding into two faces, so G has $f = f' + 1$ faces. That means that

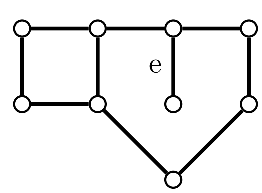
$$n - (k+1) + (f'+1) = n - k + f' = 2$$

which is the desired result.

Corollary. Let G be a connected planar graph, and let f and f' be the number of faces in two different crossing-free embeddings of G. Then $f = f'$.

Proof. From Euler's Formula, $f = 2 + m - n$ and $f' = 2 + m - n$, so $f = f'$.

Definition. We define the *degree* of a face in a crossing-free embedding to be the number of edge "sides" that the face is touching. For example, in the below graph the degree of the right enclosed face is 8, because both sides of e touch G.



Example. Let $G = (V, E)$ be a planar graph. The sum of the degrees of all of the faces in any crossing-free embedding of G is $2|E|$.

Proof. Each edge in the graph either lies on the border between two faces or lies entirely within one face. In either case, both sides of the edge touch one of the faces, so each edge contributes two to the sum of degrees of the faces. The claim follows.

Using the above definition and proposition, we can prove some other corollaries to Euler's formula:

Corollary. Let $G = (V, E)$ be a planar graph with at least two edges. Then $|E| \leq 3|V| - 6$.

Proof. If G is not connected, label the connected components of G $C_1, \ldots, C_k$, and for each $i \in [1..k-1]$, add an edge between some vertex of C_i and some vertex of C_{i+1}. Let the resulting graph be $G' = (V, E')$. Because we added edges, $|E'| \geq E$.

G' is still planar (see if you can convince yourself why), so we can fix a crossing-free embedding $\mathcal{E}$ for it. By Euler's formula $f = 2 + |E'| - |V|$. If we consider the total degree of all of the faces in $\mathcal{E}$, from our above result it is equal to $2|E'|$. On the other hand, if G' is acyclic then $f = 1$ and the degree of the one face is at least $4 \geq 3f$ (because we assumed that G had at least two edges). Otherwise, there is a cycle on each face's border, and cycles have at least 3 edges, so the total degree is at least $3f$.

Thus,

$$2|E'| \geq 3f = 3 \cdot (2 + |E'| - |V|) = 6 + 3|E'| - 3|V|$$

Re-arranging, $|E'| \leq 3|V| - 6$, so $|E| \leq 3|V| - 6$.