Mathematical Foundations of Computer Science
Lecture Outline
November 12, 2024

Equivalence Relations

A relation R on a set A is an *equivalence relation* if and only if it is reflexive, symmetric and transitive.

Example Let m be a positive integer. Show that the *congruent modulo m* relation

$$R = \{(a, b) : a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.
(If m is a positive integer then integers x and y are *congruent modulo m*, written as $x \equiv y \pmod{m}$, if $m|(x - y)$).

Solution. To show that R is an equivalence relation we need to show that it is reflexive, symmetric, and transitive. R is reflexive because $a - a = 0$, and $0 = m \cdot 0$. R is symmetric because if $a \equiv b \pmod{m}$, it means that $a - b = m \cdot k$, for some integer k. Thus $b - a = m(-k)$ and hence $(b, a) \in R$. To show that R is transitive, suppose that that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Thus, for some integers q_1 and q_2, we have $a - b = m(q_1)$ and $b - c = m(q_2)$. Adding these two equations, we get $a - c = m(q_1 + q_2)$ and thus $a \equiv c \pmod{m}$. Hence R is transitive.

Example. Suppose that R is the relation on the set of strings of English letters such that $a\,R\,b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution. R is reflexive as $l(a) = l(a)$, for any string a, and hence $a\,R\,a$. Next, suppose that $a\,R\,b$. This means that $l(a) = l(b)$ and hence $l(b) = l(a)$. Thus $b\,R\,a$ and hence R is symmetric. Finally, suppose that $a\,R\,b$ and $b\,R\,c$. Thus $l(a) = l(b)$ and $l(b) = l(c)$, which implies that $l(a) = l(c)$. Hence $a\,R\,c$ and R is transitive. Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Equivalence Classes

Let R be an equivalence relation on a set A and let $a \in A$. The *equivalence class of a*, denoted by $[a]_R$[1], is the set of all elements of A related (by R) to a; that is

$$[a]_R = \{x \in A \,|\, a\,R\,x\}$$

If $b \in [a]_R$, then b is called the *representative* of this equivalence class. Any element in a class can be used as a representative of the class.

[1]The subscript R in $[a]_R$ is dropped when the relation in reference is clear from the context.

Example. Let R be an equivalence relation on a set A. Then the following statements for elements $a, b \in A$ are equivalent

(i) $b \in [a]$ (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Solution. We will prove (i) $\implies$ (ii), (ii) $\implies$ (iii), and (iii) $\implies$ (i).
(i) $\implies$ (ii): We will prove the claim by showing that when $b \in [a]$, $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Let c be any arbitrary but particular element in $[a]$. By definition, $a\,R\,c$. Since $b \in [a]$, it means that $a\,R\,b$, which further implies $b\,R\,a$ (since R is symmetric). Since R is transitive and we know that $b\,R\,a$ and $a\,R\,c$, we have $b\,R\,c$ and thus $c \in [b]$. We have thus proved that $[a] \subseteq [b]$.

Let $d \in [b]$. By definition, $b\,R\,d$. We also know that $a\,R\,b$. Since R is transitive, $a\,R\,b$ and $b\,R\,d$, we have $a\,R\,d$. Thus, by definition, $d \in [a]$. We have thus proved that $[b] \subseteq [a]$.

(ii) $\implies$ (iii): To prove this we just need to show that $[a] \neq \emptyset$. Since R is reflexive, we know that $a \in [a]$. Since $[a] = [b]$ and $[a]$ is non-empty, it follows that $[a] \cap [b] \neq \emptyset$.

(iii) $\implies$ (i): Let $c \in [a] \cap [b]$. Thus $a\,R\,c$ and $b\,R\,c$. Since R is symmetric, we have $c\,R\,b$. Since R is transitive, $a\,R\,c$ and $c\,R\,b$, we have $a\,R\,b$. By, definition $b \in [a]$.

Example. Let R be an equivalence relation on a set A. Then the set $\{[a]_R \mid a \in A\}$ is a partition of the set A. Each element of the set is called an *equivalence class* of R. Conversely, given a partition $\{A_i\}$ of the set A, there is an equivalence relation R that has sets A_i as its equivalence classes.

Solution. Since each element $a \in A$ is in its own equivalent class $[a]$, each equivalent class is non-empty and $\bigcup_{a \in A}[a] = A$. From the claim in the previous example, for any two elements a and b in A, $[a]$ and $[b]$ are either equal or disjoint. Thus the equivalent classes partition the set A.

We now prove the converse. Let R be the relation on A that contains all possible pairs (x, y), where x and y belong to the same subset A_i in the partition. We want to show that R is reflexive, symmetric and transitive. R is reflexive as any element $a \in A$ is in the same subset of the partition as itself. Next suppose that $a\,R\,b$. This means that a and b are in the same subset of the partition of A. Thus, we have $b\,R\,a$ and hence R is symmetric. Finally, suppose that $a\,R\,b$ and $b\,R\,c$. This means that a and b are in the same subset of the partition and so are b and c. This means that a and c are in the same subset of the partition and hence we have $a\,R\,c$. Thus R is transitive.

Example. If an equivalence relation R is defined by the following set partition on A, then express R as a set of ordered pairs.

$$A = \{3, 4, 1\} \cup \{2\}$$

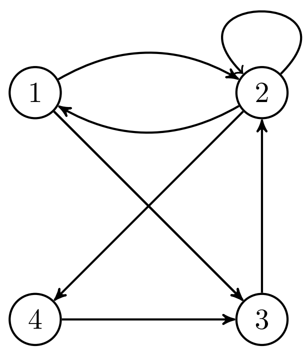
Solution.

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(3,1),(3,4),(4,3),(4,1)\}$$

Representing Relations Using Directed Graphs

A *directed graph*, or *digraph* $G = (V, E)$ consists of a set V of vertices and a subset $E \subseteq V \times V$ of edges or arcs. An edge of the form (u, u) is represented as an arc from u to itself.

A binary relation R on a set A can be represented as a directed graph in which the vertices represent the elements of A and for every ordered pair $(a, b) \in R$, there is an edge from vertex a to vertex b. For example, the digraph corresponding to the relation $R = \{(1,2),(1,3),(2,1),(2,2),(2,4),(3,2),(4,3)\}$ on the set $\{1, 2, 3, 4\}$ is shown below.



The directed graph G representing a relation R can be used to determine properties of the relation R. R is reflexive iff G contains a self-loop at every vertex. R is symmetric iff for each edge (a, b) $(a \neq b)$ in G, there is also an edge (b, a) in G. R is antisymmetric iff for any two distinct vertices a, b there are no edges between them or exactly one of (a, b) or (b, a) is in G. Thus R is antisymmetric iff for any two distinct vertices a and b, both (a, b) and (b, a) are not present in G. The relation R is transitive iff edge (u, w) always exists whenever there is an edge (u, v) and (v, w), for some vertex v.

The Probabilistic Method

A *tournament graph* is a directed graph with exactly one directed edge between any pair of vertices. Every tournament graph has at least one Hamiltonian path, a path that visits each vertex exactly once (can be proved using induction). In 1943, Szele used the Probabilistic Method to show the existence of a tournament graph with a large number of Hamiltonian paths. Note that there are tournaments in which there is exactly one Hamiltonian path. For example, the tournament on vertices $\{1, 2, \ldots, n\}$ in which there is a directed edge (i, j) iff $i < j$ has exactly one Hamiltonian path.

Example. Prove that there is a n-vertex tournament with at least $\frac{n!}{2^{n-1}}$ distinct Hamiltonian paths.

Solution. Let $G = (n, 1/2)$ be a n-vertex tournament graph, in which an edge between any two vertices u and v is directed towards u with probability $\frac{1}{2}$ and towards v with probability $\frac{1}{2}$. Let X denote the total number of Hamiltonian paths in G and let X_σ be an indicator random variable that is 1, iff a permutation σ of the vertices in G yields a Hamiltonian path. Clearly, $X = \sum_\sigma X_\sigma$. Applying the Linearity of Expectation, we get

$$\begin{aligned}
\mathbf{E}[X] &= \sum_\sigma \mathbf{E}[X_\sigma] \\
&= \sum_\sigma \Pr[X_\sigma = 1] \\
&= \sum_\sigma \left(\frac{1}{2}\right)^{n-1} \\
&= \frac{n!}{2^{n-1}}
\end{aligned}$$

Since a random orientation of the edges, i.e., a random tournament, yields us the above number in expectation, there must be an orientation of the edges, i.e., a tournament, in which the number of Hamiltonian paths is at least $n!/2^{n-1}$.