Mathematical Foundations of Computer Science
Lecture Outline
November 5, 2024

Matching in Bipartite Graphs

An *independent set* of a graph is a set of pair-wise non-adjacent vertices. A *bipartite graph*, (U, V, E), is a graph whose vertex set is $U \cup V$ and for each edge $e = (u, v) \in E$, $u \in U$ and $v \in V$. In other words, U and V are independent sets and each edge in E connects a vertex in U to a vertex in V.

Now consider the following scenario. There is a set of girls and a set of boys. Each girl likes some boys and dislikes others. What conditions would guarantee that each girl is paired-up with a boy that she likes and that no two girls are paired-up with the same boy.

We can model this situation using a bipartite graph, (X, Y, E), where each vertex in X represents a girl, each vertex in Y represents a boy and and edge $(g, b) \in E$ means that girl g likes boy b. We are interested in the conditions that would guarantee a matching that saturates every vertex in X.
Hall's theorem gives the necessary and sufficient conditions for the existence of such matchings in bipartite graphs.

Example. [Hall's Theorem] Let $G = (X, Y, E)$ be a bipartite graph. For any set S of vertices, let $N_G(S)$ be the set of vertices adjacent to vertices in S. Prove that G contains a matching that saturates every vertex in X iff $|N_G(S)| \geq |S|, \forall S \subseteq X$. The condition "For all $S \subseteq X, |N(S)| \geq |S|$" is called Hall's condition.

Solution. We prove that Hall's condition is necessary as follows. Suppose G contains a matching M that saturates every vertex in X. Let S be a subset of X. Since each vertex in S is matched under M to a distinct vertex in $N_G(S)$, $|N_G(S)| \geq |S|$.

We will now prove the sufficiency of Hall's conditon, i.e., if $|N_G(S)| \geq |S|, \forall S \subseteq X$ then G contains a matching that saturates every vertex in X. We prove this by induction on the size of X.

Base Case: $|X| = 1$. If the only vertex in X is connected to at least one vertex in Y then clearly a matching exists.
Induction Hypothesis: Assume that Hall's condition is sufficient when $|X| = j$, for all j such that $1 \leq j \leq k$.
Induction Step: We want to prove that the sufficiency of Hall's condition when $|X| = k+1$. Let $G = (X, Y, E)$ be a graph with $k+1$ vertices in X such that $\forall S \subseteq X, |N_G(S)| \geq |S|$.

We consider the following two cases.
Case I: For every non-empty proper subset $W \subset X$, $|N_G(W)| > |W|$. In this case, we pair-up an arbitrary vertex $x \in X$ with one of its neighbors, say $y \in Y$. Now consider the subgraph $G' = (X', Y', E')$, where $X' = X \setminus \{x\}$, $Y' = Y \setminus \{y\}$, and $E' = E \setminus \{(x, y)\})$. After the removal of y, the neighborhood of any subset, $S' \subseteq X'$ in G' is at most one less than its neighborhood in G. But since $|N_G(S')| > |S'|$, after removal of y, it must be that $|N_{G'}(S')| \geq |S'|$. Thus, Hall's condition holds for G'. By induction hypothesis, G' contains a matching M' that saturates every vertex in X'. Hence, $M' \cup \{(x, y)\}$ is a matching that saturates every vertex in X.
Case II: For some non-empty proper subset $W \subset X$, $|N(W)| = |W|$. For all $S' \subseteq W$, we have $N_G(S') \subseteq N_G(W)$. Hence, Hall's condition holds for the subgraph induced by $W \cup N(W)$. By induction hypothesis, there is a matching M_1 that matches every vertex in W to a vertex in $N_G(W)$. Note that M_1 is a perfect matching. Consider the subgraph $G' = (X', Y', E')$, where $X' = X \setminus W$, $Y' = Y \setminus N(W)$, and E' consists of all edges between X' and Y'. If we can prove that Hall's condition holds for G' then by induction hypothesis, G' has a matching M_2 that saturates every vertex in X'. Then, $M_1 \cup M_2$ is clearly a matching in G that saturates every vertex in X. It now remains to prove that $\forall T \subseteq X', |N_{G'}(T)| \geq |T|$. Note that $N_G(W \cup T) = N_G(W) \cup N_{G'}(T)$, $|N_G(W)| = |W|$, W and T are disjoint, and $N_G(W)$ and $N_{G'}(T)$ are disjoint. Then,

$$
\begin{aligned}
|N_G(W \cup T)| &\geq |W \cup T| \text{ (follows because } \forall S \subseteq X, |N_G(S)| \geq |S|) \\
|N_G(W)| + |N_{G'}(T)| &\geq |W| + |T| \\
|W| + |N_{G'}(T)| &\geq |W| + |T| \\
|N_{G'}(T)| &\geq |T|
\end{aligned}
$$

This proves the sufficiency of Hall's condition.

Relations

A *binary relation* is a set of ordered pairs. For example, let $R = \{(1, 2), (2, 3), (5, 4)\}$. Then since $(1, 2) \in R$, we say that 1 is related to 2 by relation R. We denote this by $1\,R\,2$. Similarly, since $(4, 7) \notin R$, 4 is not related to 7 by relation R, denoted by $4 \not R\ 7$.

A binary relation R from set A to set B is a subset of the cartesian product $A \times B$. When $A = B$, we say that R is a relation on set A.

Example. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Consider the following relations.

$$
\begin{aligned}
R_1 &= \{(1, 1), (1, 2), (2, 2), (2, 3)\} \\
R_2 &= \{(1, 2), (2, 3), (3, 4), (4, 1), (4, 4)\} \\
R_3 &= \{(1, a), (2, a), (3, b), (4, c)\} \\
R_4 &= \{(a, 1), (a, 3), (a, 4), (c, 1)\} \\
R_5 &= \{(a, a), (a, b), (1, c)\}
\end{aligned}
$$

R_1 and R_2 are relations on A. R_3 is a relation from A to B. R_4 is a relation from B to A. R_5 is not a relation on sets A and B and it is neither a relation from A to B nor a relation from B to A. It is a relation on $A \cup B$.

Below are some more examples of relations.

- If S is a set then "is a subset of ", $\subseteq$ is a relation on $\mathcal{P}(S)$, the power set of S.
- "is a student in" is a relation from the set of students to the set of courses.
- "=" is a relation on $\mathbb{Z}$.
- "has a path in G to" is a relation on $V(G)$, the set of vertices in G.

Example. How many relations are there on a set A of n elements?

Solution. Note that any relation on A is a subset of $A \times A$ and since the power set of $A \times A$ contains all subsets of $A \times A$, the number of possible relations on A is the cardinality of the power set of $A \times A$. Since $|A \times A| = n^2$, the cardinality of the power set of $A \times A$ is 2^{n^2}. Thus our answer is 2^{n^2}.

Properties of Relations

Let R be a relation defined on set A. We say that R is

- *reflexive*, if for all $x \in A$, $(x, x) \in R$.
- *irreflexive*, if for all $x \in A$, $(x, x) \notin R$.
- *symmetric*, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- *antisymmetric*, if for all $x, y \in A$, $x\,R\,y$ and $y\,R\,x \implies x = y$.
- *transitive*, if for all $x, y, z \in A$, $x\,R\,y$ and $y\,R\,z \implies x\,R\,z$.

Note that the terms *symmetric* and *antisymmetric* are not opposites. A relation may be both symmetric and antisymmetric or can neither be symmetric nor be antisymmetric.

Example. What are the properties of the following relations?

R_1 : equality relation on $\mathbb{Z}$.
R_2 : "is a sibling of" relation on the set of all people.
R_3 : " $\leq$ " relation on $\mathbb{Z}$.
R_4 : " $<$ " relation on $\mathbb{Z}$.
R_5 : "|" relation on $\mathbb{Z}^+$.
R_6 : "|" relation on $\mathbb{Z}$.
R_7 : " $\subseteq$ " relation on the power set of a set S.
$R_8 : \{(x, y) \in \mathbb{R}^2 \,:\, |x - y| < \epsilon\},\;$ where $\epsilon = 0.001$

Solution.

$$\begin{aligned} \text{Reflexive} &: R_1, R_3, R_5, R_7, R_8 \\ \text{Irreflexive} &: R_2, R_4 \\ \text{Symmetric} &: R_1, R_2, R_8 \\ \text{Antisymmetric} &: R_1, R_3, R_4, R_5, R_7 \\ \text{Transitive} &: R_1, R_3, R_4, R_5, R_6, R_7 \end{aligned}$$

Note that R_6 is not reflexive because $(0,0) \notin R_6$; it is not antisymmetric because for any integer a, $a | -a$ and $-a | a$, but $a \neq -a$. R_2 is not transitive because x and z could be the same person. Observe that R_6 is an example of a relation that is neither symmetric nor antisymmetric. R_1 is an example of a relation that is symmetric and antisymmetric.