Mathematical Foundations of Computer Science
Lecture Outline
November 7, 2024

The Geometric Distribution

Consider the following question. Suppose we have a biased coin with heads probability p that we flip repeatedly until it lands on heads. What is the distribution of the number of flips? This is an example of a *geometric distribution*. It arises in situations where we perform a sequence of independent trials until the first success where each trial succeeds with a probability p.

Note that the sample space Ω consists of all sequences that end in H and have exactly one H. That is

$$\Omega = \{H, TH, TTH, TTTH, TTTTH, \ldots\}$$

For any $\omega \in \Omega$ of length i, $\Pr[\omega] = (1-p)^{i-1}p$.

Definition. A *geometric random variable* X with parameter p is given by the following distribution for $i = 1, 2, \ldots$:

$$\Pr[X = i] = (1-p)^{i-1}p$$

We can verify that the geometric random variable admits a valid probability distribution as follows:

$$\sum_{i=1}^{\infty}(1-p)^{i-1}p = p\sum_{i=1}^{\infty}(1-p)^{i-1} = \frac{p}{1-p}\sum_{i=1}^{\infty}(1-p)^{i} = \frac{p}{1-p}\cdot\frac{1-p}{1-(1-p)} = 1$$

Note that to obtain the second-last term we have used the fact that $\sum_{i=1}^{\infty} c^i = \frac{c}{1-c}$, $|c| < 1$.

Let's now calculate the expectation of a geometric random variable, X. We can do this in several ways. One way is to use the definition of expectation.

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{i=0}^{\infty} i \Pr[X = i] \\
&= \sum_{i=0}^{\infty} i(1-p)^{i-1}p \\
&= \frac{p}{1-p}\sum_{i=0}^{\infty} i(1-p)^{i} \\
&= \left(\frac{p}{1-p}\right)\left(\frac{1-p}{(1-(1-p))^2}\right) \quad \left(\because \sum_{i=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \text{ for } |x| < 1.\right) \\
&= \left(\frac{p}{1-p}\right)\left(\frac{1-p}{p^2}\right) \\
&= \frac{1}{p}
\end{aligned}$$

Another way to compute the expectation is to note that X is a random variable that takes on non-negative integer values. From a theorem proved earlier we know that if X takes on only non-negative integer values then

$$\mathbf{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

Using this result we can calculate the expectation of the geometric random variable X. For the geometric random variable X with parameter p,

$$\Pr[X \geq i] = \sum_{j=i}^{\infty}(1-p)^{j-1}p = (1-p)^{i-1}p\sum_{j=0}^{\infty}(1-p)^{j} = (1-p)^{i-1}p \times \frac{1}{1-(1-p)} = (1-p)^{i-1}$$

Therefore

$$\mathbf{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] = \sum_{i=1}^{\infty}(1-p)^{i-1} = \frac{1}{1-p}\sum_{i=1}^{\infty}(1-p)^{i} = \frac{1}{1-p} \cdot \frac{1-p}{1-(1-p)} = \frac{1}{p}$$

Memoryless Property. For a geometric random variable X with parameter p and for $n > 0$,

$$\Pr[X = n + k \mid X > k] = \Pr[X = n]$$

Solution.

$$\begin{aligned}\Pr[X = n+k \mid X > k] &= \frac{\Pr[X = n+k \cap X > k]}{\Pr[X > k]} \\ &= \frac{\Pr[X = n+k]}{\Pr[X > k]} \\ &= \frac{(1-p)^{n+k-1}p}{(1-p)^k} \\ &= (1-p)^{n-1}p \\ &= \Pr[X = n]\end{aligned}$$

Definition: Conditional Expectation. The following is the definition of conditional expectation.

$$\mathbf{E}[Y \mid Z = z] = \sum_y y \Pr[Y = y \mid Z = z],$$

where the summation is over all possible values y that the random variable Y can assume.

Example. For any random variables X and Y,

$$\mathbf{E}[X] = \sum_y \Pr[Y = y]\mathbf{E}[X \mid Y = y]$$

Solution.

$$\begin{aligned}\mathbf{E}[X] &= \sum_x x \cdot \Pr[X = x] \\ &= \sum_x x \cdot \sum_y \Pr[X = x \cap Y = y] && \text{(By Law of Total Probability)} \\ &= \sum_x x \cdot \sum_y (\Pr[X = x|Y = y] \cdot \Pr[Y = y]) \\ &= \sum_y \Pr[Y = y] \cdot \sum_x (x \cdot \Pr[X = x|Y = y]) \\ &= \sum_y \Pr[Y = y] \cdot \mathbf{E}[X|Y = y]\end{aligned}$$

We can also calculate the expectation of a geometric random variable X using the memoryless property of the geometric random variable. Let Y be a random variable that is 0, if the first flip results in tails and that is 1, if the first flip is a heads. Using conditional expectation we have

$$\begin{aligned}\mathbf{E}[X] &= \Pr[Y = 0]\mathbf{E}[X|Y = 0] + \Pr[Y = 1]\mathbf{E}[X|Y = 1] \\ &= (1-p)(\mathbf{E}[X] + 1) + p \cdot 1 \qquad \text{(using the memoryless property)} \\ \therefore p\mathbf{E}[X] &= 1 \\ \mathbf{E}[X] &= \frac{1}{p}\end{aligned}$$

Binomial Distributions

Consider an experiment in which we perform a sequence of n coin flips in which the probability of obtaining heads is p. How many flips result in heads?

If X denotes the number of heads that appear then

$$\Pr[X = j] = \binom{n}{j} p^j (1-p)^{n-j}$$

Definition. A *binomial* random variable X with parameters n and p is defined by the following probability distribution on $j = 0, 1, 2, \ldots, n$:

$$\Pr[X = j] = \binom{n}{j} p^j (1-p)^{n-j}$$

We can verify that the above is a valid probability distribution using the binomial theorem as follows

$$\sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} = (p + (1-p))^n = 1$$

What is the expectation of a binomial random variable X? We can calculate $\mathbf{E}[X]$ is two ways. We first calculate it directly from the definition.

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{j=0}^{n} j \binom{n}{j} p^j (1-p)^{n-j} \\
&= \sum_{j=0}^{n} j \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
&= \sum_{j=1}^{n} j \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
&= \sum_{j=1}^{n} \frac{n!}{(j-1)!(n-j)!} p^j (1-p)^{n-j} \\
&= np \sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!((n-1)-(j-1))!} p^{j-1} (1-p)^{(n-1)-(j-1)} \\
&= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!} p^k (1-p)^{(n-1)-k} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} \\
&= np
\end{aligned}$$

The last equation follows from the binomial expansion of $(p + (1-p))^{n-1}$.

We can obtain the result in a much simpler way by using the linearity of expectation. Let $X_i, 1 \leq i \leq n$ be the indicator random variable that is 1 if the ith flip results in heads and is 0 otherwise. We have $X = \sum_{i=1}^{n} X_i$. By the lineartity of expectation we have

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} p = np$$

What is the variance of the binomial random variable X? Since $X = \sum_{i=1}^{n} X_i$, and $X_1, X_2, \ldots, X_n$ are independent we have

$$\begin{aligned}
\text{Var}[X] &= \sum_{i=1}^{n} \text{Var}[X_i] \\
&= \sum_{i=1}^{n} \mathbf{E}[X_i^2] - \mathbf{E}[X_i]^2 \\
&= \sum_{i=1}^{n} (p - p^2) \\
&= np(1-p)
\end{aligned}$$

Coupon Collector's Problem.

We are trying to collect n different coupons that can be obtained by buying cereal boxes. The objective is to collect at least one coupon of each of the n types. Assume that each cereal box contains exactly one coupon and any of the n coupons is equally likely to occur. How many cereal boxes do we expect to buy to collect at least one coupon of each type?

Solution. Let the random variable X denote the number of cereal boxes bought until we have at least one coupon of each type. We want to compute $\mathbf{E}[X]$. Let X_i be the random variable denoting the number of boxes bought to get the ith new coupon. Clearly,

$$X = X_1 + X_2 + X_3 + \ldots + X_n$$

Using the linearity of expectation we have

$$\mathbf{E}[X] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \mathbf{E}[X_3] + \ldots + \mathbf{E}[X_n] \tag{1}$$

What is the distribution of random variable X_i? Observe that the probability of obtaining the ith new coupon is given by

$$p_i = \frac{n - (i-1)}{n} = \frac{n - i + 1}{n}$$

Thus the random variable $X_i, 1 \leq i \leq n$ is a geometric random variable with parameter p_i.

$$\mathbf{E}[X_i] = \frac{1}{p_i} = \frac{n}{n - i + 1}$$

Combining this with equation (1) we get

$$\mathbf{E}[X] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{2} + \frac{n}{1} = n\sum_{i=1}^{n}\frac{1}{i}$$

The summation $\sum_{i=1}^{n}\frac{1}{i}$ is known as the *harmonic number* $H(n)$ and $H(n) = \ln n + c$, for some constant $c < 1$.

Hence the expected number of boxes needed to collect n coupons is about $nH(n) < n(\ln n + 1)$.

Example. How many reflexive relations are there on a set A of size n?

Solution. We know that $R \subseteq A \times A$. The procedure of constructing a reflexive relation R is as follows:

> Step 1: From $A \times A$, include in R all ordered pairs of the form (a, a).
> Step 2: For every ordered pair in $A \times A$ of the form (a, b), where $a \neq b$, choose whether to include it in R or not.

There is one way to do Step 1 and $2^{n(n-1)}$ ways to do Step 2. By the multiplication rule, the number of reflexive relations on a set n elements is $2^{n(n-1)}$.

Equivalence Relations

A relation R on a set A is an *equivalence relation* if and only if it is reflexive, symmetric and transitive.