CIS 1600 — Assigned: October 23, 2024

Homework 9H

Due: 11:59PM EDT, October 28, 2024

This homework is due electronically on Gradescope at 11:59PM EDT, October 28, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. LaTeX: All solutions are required to be typeset in LaTeX.

B. Standard Deductions:
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: Please make sure to strictly follow our collaboration policy as clarified on Ed.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

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1. [12 pts] Sid's As-Sid-duous Search

Sid is obsessed with collecting the yellow leaves that fall from autumn trees. To be able to maximize his leaf-collecting abilities, he has created a map of all the parks this side of the Schuylkill, where at most one bidirectional trail exists between each pair of parks. Upon studying his map before going on his leaf-collecting expedition, Sid finds that he is able to reach any park from any other park by following a series of trails on the map. Given that each park has an even number of trails extending from it, prove that Sid's finding holds true even if one of the trails is under maintenance and cannot be used.

We strongly encourage you to approach this problem without induction (our solution does not use induction). This is not to say it is not possible (correct proofs will receive credit), but induction proofs are not always so simple and it's important to learn how to construct graph proofs using non-inductive methods as well.

2. [12 pts] Corn Conundrums

Winston wants to buy corn for his annual fall harvest picnic and goes to his local farm to purchase it. The $n > 5$ cornstalks at the farm each have a height of x_i, where each x_i is a distinct value in $[1, ..., n]$. He decides to pick cornstalks randomly without replacement, stopping when the cornstalks are no longer decreasing in height; in other words, he picks cornstalks with heights $x_1, x_2, \ldots$ until he picks a cornstalk with height x_k where $x_1 > x_2 > x_3 > \ldots > x_{k-1}$ and $x_{k-1} < x_k$. Since Winston loves corn, he also includes the kth cornstalk in his batch.

(a) What is the probability Winston leaves with at least 5 corn?

(b) What is the probability Winston leaves with exactly 5 corn?

3. [10 pts] LETS JUMP! LEAVES LEAVES LEAVES!

E-leaf-abeth decides to brings her own set of leaves for all her friends at Penn to make a leaf pile to play in! Her friends line up to start picking up leaves. She knows that 90% of the leaves on the ground are red. Each red leaf has a 1/20 chance of being a friend's favorite type of leaf in the pile when they pick it up, independent of whether or not the leaf was a favorite for any other friend. The remaining 10% of leaves are green (a rare and unappreciated color), so they have a 1/1000 chance of being a friend's favorite leaf once they pick it up, again independent of the opinions of any other friends.

E-leaf-abeth chooses to set apart one leaf for all the other friends to pick up before the playdate starts. But, plot twist, she's colorblind, so she can't figure out if it was a red or green leaf. She randomly sets aside the leaf on the ground. Let X be the random variable denoting the number of friends in line that pick up the leaf until it becomes some friend's favorite leaf.

When E-leaf-abeth goes back home, she organizes a playdate with her hometown friends, but she brings the same set of leaves as the first playdate. She once again initially sets aside the same exact leaf before her friends arrive. Assume that this leaf has the same probability of being a friend's favorite as the corresponding leaf from the first playdate. Let Y be the random variable denoting the number of hometown friends in line that pick up the leaf until this leaf is a friend's favorite in this second playdate.

Are the random variables X and Y independent? For this question, you do not need to define a sample space, but you must define any other events or random variables you may use.

4. [12 pts] Incoming Business Analyst @ McKinsey

One brisk fall afternoon, the venerable Suzzy is sitting pensively upon the ground, frolicking within a pile of n leaves when she spontaneously decides to simulate an imaginary networking session between the leaves. After a grueling afternoon of simulating conversations, she ensures that by the end of the session, any two leaves have talked with each other with probability $\frac{1}{2}$. What is the expected number of unique groups of three leaves that Suzzy can find within all the n leaves such that all three leaves in that group have talked to each other?

5. [12 pts] YYYYYYYYYYOOOOOOOO

On a lovely Fall day, Megan walks past her favorite tree on Locust. As she walks towards the tree, 10 yellow leaves and 8 orange leaves fall off the tree and form a straight line on the ground uniformly at random. How many pairs of consecutive leaves, where the leaves in the pair are a different color, can Megan expect to see?

For example, if the leaves formed the line YOOOYYYYYYYYOOOOYO (where Y is yellow and O is orange), there would be 5 such pairs. Note that the last two pairs overlap.

6. [12 pts] A (Red) Delicious Drive

Sophia Fu-ji and Kevin Han-eycrisp are going on an apple picking expedition. However, on the car ride there, they get into a fight about the solution to a CIS 1600 problem and become bitter enemies. Determined to still pick as many apples as possible, they devise a plan. Every pair of apple trees on the farm has exactly one road between them, either paved or unpaved. Note that there is no restriction on which apple trees they can visit.

Sophia will only take the paved roads and notices that the set of apple trees V and the set of paved roads R form a simple, undirected graph $G = (V, R)$. Meanwhile, Kevin sets out to chart his own course, traveling between apple trees only when the road between them is unpaved. Let the graph representing the set of apple trees and set of unpaved roads Kevin is assigned is $\overline{G} = (V, \overline{R})$, where

$$\overline{R} = \{\{x, y\} \mid x \neq y, \{x, y\} \notin R\}$$

Define a *cut vertex* to be a vertex such that the subgraph obtained by deleting it and all of its incident edges has more connected components than the original graph.

Prove that if a vertex v is a cut vertex in G, then v is not a cut vertex in $\overline{G}$.