This homework is due electronically on Gradescope at 11:59PM EDT, October 21, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. LaTeX: Please typeset all your answers in LaTeX based on the template we provide for you. Failure to do so will result in a 0 for the homework.

B. Standard Deductions:
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: Please make sure to strictly follow our collaboration policy as clarified on Ed.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. **[14 pts] Rolling for Riches**

   Andrew loves Genshin Impact and is trying to increase his count of primogems. Andrew tests the in-game "wish" system using his primogems.

   Whenever Andrew enters the wish menu where he may choose to wish, behind the scenes and before the game reveals what Andrew may get, the game tosses a (virtual) fair coin labelled 1 on one side and 2 on the other. The game then rolls a fair 40 sided die.

   Andrew now has the option to pay 20 primogems to wish.

   He is given the option between choosing a normal wish, in which the game looks at the coin toss first, and a "special wish" in which the game looks at the dice roll first. If the result of whichever he chooses is 1, then he will win 80 primogems. If the result is anything else, Andrew will lose 32 additional primogems and the wish ends there.

   If Andrew wins, then in the second round of the process, the game will reveal what Andrew rolled on the object he did not choose. If the result of the second object is a 1, he will win 800 more primogems! However, if it is any other number, he will lose 32 primogems.

   Andrew isn't sure whether it is worth it to continue wishing, and in order to maximize his potential of obtaining as many virtual characters for his team as possible, he asks the following questions:

   (a) Let X be a random variable that denotes the result of the coin flip. What is $E[X]$?

   (b) Let Y be a random variable that denotes the result of the die roll. What is $E[Y]$?

   (c) If Andrew decides to not to wish, what is the expected value of his primogem earnings?

   (d) If Andrew decides to play normal wish (look at the coin first), what is the expected value of his primogem earnings?

   (e) If Andrew decides to special wish (look at the die first), what is the expected value of his primogem earnings?

2. **[10 pts] Minecraft Madness**

   When head TAs Andrew, Cindy, Eric, and Dilini aren't creating new homework problems or in meetings with Rajiv, they like playing Minecraft. After pulling an all-nighter playing Minecraft, Andrew and Cindy have obtained 1600 diamonds each. Andrew labels his diamonds 1 through 1600, while Cindy labels her diamonds 1 through 1600. Meanwhile, Eric and Dilini have obtained 1600 emeralds each. Eric labels his emeralds 1 through 1600, while Dilini labels her emeralds 1 through 1600.

   To tell their minerals apart, Andrew and Eric put all of their minerals into one chest, while Cindy

and Dilini put their minerals into another one (there are 1600 diamonds and 1600 emeralds in *each* chest). Now, we pick one mineral uniformly at random from each chest.

(a) Given that at least one of the two minerals selected is a diamond, what is the probability that both minerals chosen are diamonds?

(b) Given that at least one of the two minerals selected is a diamond labeled with 37, what is the probability that both minerals are diamonds?

3. **[10 pts] Clan Wars**

Max (Town Hall 15) loves playing Clash of Clans. It's clan war season and Max, who is in a clan of 6 people, is going up against Daniel, who is also in a clan of 6 people. Each person in both clans independently conducts 1 raid with a $\frac{1}{2}$ probability of winning. Furthermore, Daniel is so bad that his raid is guaranteed to lose (i.e. Max's clan will conduct 6 potentially successful raids, and Daniel's clan will conduct only 5 potentially successful raids). What is the probability that Max's clan has strictly more winning raids than Daniel's?

4. **[12 pts] PvZ Water Map**

Plants vs. Zombies just introduced a new map that is completely water-based. You, the player, have been easily blitzing through the stages until you reach the bonus 5th stage. Here, the only plants you have available are lily pads and bamboos, which can be deployed directly on water. A bamboo connects exactly two lily pads and each pair of lily pads is connected by at most one bamboo. In order to win, you must form a super pad, which can decimate any zombie that appears on the map in one shot. A super pad is formed in any configuration containing a lily pad connected to at least two different lily *leaves* (lily leaves are lily pads connected to exactly one other lily pad).

Crazy David suddenly appears and decides to take control over the garden. He deploys the lily pads and bamboos such that the graph spanned by the lily pads and bamboos forms a tree. He observes every lily pad connected to a lily leaf is connected to at least two other lily pads or leaves. Prove that a super pad will always exist.

5. **[12 pts] Suspicious Spaceship**

The CIS 1600 TAs are aboard a spaceship, the Skeld, traveling through outer space! The Skeld has $n \geq 3$ rooms, each containing a task for the crewmates to complete. These rooms are connected by hallways, where each hallway links exactly two rooms and can be traversed in either direction. Each pair of rooms is connected by at most one hallway. Importantly, any room can be reached from any other room by traveling through a series of hallways.

Driven by his love for optimization problems - and his suspicion of his fellow TAs - Alex Yang wants to find the best route to complete his tasks as quickly as possible. His plan is to start in

one room, travel through a nonempty sequence of rooms (without revisiting any of them), and return to the starting room.

Two routes are considered the same if one is simply a rotation or reversal of the other. That is, traveling through the same sequence of rooms and hallways in either direction counts as the same route. Help Alex complete his tasks and prove that exactly one route exists if and only if there are exactly n hallways.

6. **[12 pts] Brawl or Brain?**

   Olivia has been dying to get back to playing Brawl Stars after a grueling week of midterms, but her friends aren't very keen on joining her. Seeing her play all alone, Daniel and Luna (tunamoon) agree to join her on the condition that she won't lose all her CIS 1600 knowledge if she starts playing again. If Olivia can show she's still capable of thinking about things other than Brawl Stars, Daniel and Luna will agree to play with her. Help Olivia prove the following claim so she can pursue her passion!

   Let T be a tree with $n > 1$ vertices. Prove that the number of leaves in T must be:

   $$2 + \sum_{\substack{v_i \in V, \\ \deg(v_i) \geq 3}} (\deg(v_i) - 2)$$