CIS 1600

Homework 5T

Assigned: September 23, 2024

Due: 11:59PM EDT, September 25, 2024

This homework is due electronically on Gradescope at 11:59PM EDT, September 25, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. LaTeX: Please typeset all your answers in LaTeX based on the template we provide for you. Failure to do so will result in a 0 for the homework.

B. Standard Deductions:
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: You may not collaborate with anyone via any means.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [10 pts] The Inductive Rescue Mission

It's Friday night, and the CIS 1600 TAs are locked in an epic showdown of Egyptian Ratscrew. Unfortunately, David is down on his luck and has only one card remaining. Meanwhile, with his superb reflexes and pattern recognition skills, Eric is sitting comfortably atop a mountain of cards. Being the kind Head TA that he is, Eric promises to donate half his cards to David if he can identify whether each of the following induction proofs is valid or invalid.

Unfortunately, David's hand is much too sore for him to focus on induction. Help David stay in the game by determining whether each of the "proofs" below is valid or invalid! If it is invalid, indicate clearly as to where the logical error in the proof lies and justify why this is a logical error. If the proof is valid, you can simply say so. Just stating that the claim is false will not be awarded credit.

(a) **Claim:** $\forall n \in \mathbb{Z}^+, n^2 \leq n$.

Proof:

Base Case: For $n = 1$, the claim is true since $1^2 = 1$.

Induction Hypothesis: Assume that $k^2 \leq k$, for some $k \in \mathbb{Z}^+$.

Induction Step: We need to show that

$$(k+1)^2 \leq k+1$$

We can see that

$$k^2 \leq k^2 + 2k = (k^2 + 2k + 1) - 1 = (k+1)^2 - 1$$

Since $(k+1)^2 \leq k+1$,

$$(k+1)^2 - 1 \leq (k+1) - 1 = k$$

Thus we get $k^2 \leq k$, which we know is true by the induction hypothesis.

(b) **Claim:** $\forall n \in \mathbb{Z}^+, 5^n = 5$.

Proof: We will prove the claim using strong induction on n.

Base Case: For $n = 1$, the claim is true since $5^1 = 5$.

Induction Hypothesis: Assume that $5^j = 5$, for all integers j s.t. $1 \leq j \leq k$ for some $k \in \mathbb{Z}^+$.

Induction Step: We need to prove that $5^{k+1} = 5$, and we have

$$\begin{aligned} 5^{k+1} &= \frac{5^k 5^k}{5^{k-1}} \\ &= \frac{5 \cdot 5}{5} \quad \text{(using induction hypothesis)} \\ &= 5 \end{aligned}$$

This completes the strong induction proof, so $5^n = 5$, for all $n \in \mathbb{Z}^+$.

(c) **Claim:** For all negative integers n,

$$(-2) + (-4) + \ldots + (2n) = -n^2 + n$$

Proof: We will prove the claim using induction on n.

Base Case: The claim holds when $n = -1$ since $-2 = -(-1)^2 + (-1) = -1 - 1 = -2$.

Induction Hypothesis: Assume that $(-2) + (-4) + \ldots + (2k) = -k^2 + k$, for some $k \in \mathbb{Z}$, $k \leq -1$.

Induction Step: We want to prove that the claim is true when $n = k - 1$. That is, we want to prove that

$$(-2) + (-4) + \ldots + (2k) + (2(k-1)) = -(k-1)^2 + (k-1)$$

$$\begin{aligned}
\text{L.H.S.} &= (-2) + (-4) + \ldots + (2k) + (2(k-1)) \\
&= (-2) + (-4) + \ldots + (2k) + (2k-2) \\
&= -k^2 + k + (2k-2) && \text{(using induction hypothesis)} \\
&= -k^2 + 3k - 2 \\
&= (-k^2 + 2k - 1) + (k-1) \\
&= -(k-1)^2 + (k-1)
\end{aligned}$$

This completes the induction proof.

(d) **Claim:** $\forall n \in \mathbb{N}$, $5n = 0$.

Proof: We will prove the claim using strong induction on n.

Base Case: The claim holds when $n = 0$ since $5 \cdot 0 = 0$.

Induction Hypothesis: Assume that $5j = 0$, for all $0 \leq j \leq k$, for some $k \in \mathbb{N}$, $j \in \mathbb{Z}$.

Induction Step: We must show that $5(k+1) = 0$. Let $k + 1 = a + b$, where $a > 0$ and $0 < b \leq k$ are integers. From the induction hypothesis we know that $5a = 0$ and $5b = 0$, therefore

$$5(k+1) = 5(a+b) = 5a + 5b = 0 + 0 = 0$$

This completes the induction proof.

(e) **Claim:** $\forall n \in \mathbb{Z}^+$, if p and q are positive integers such that $\max(p, q) = n$, then $p = q$.

Proof:

Base Case: Let $n = 1$. If p and q are positive integers such that $\max(p, q) = 1$, then p and q must both be 1, satisfying the claim.

Induction Hypothesis: Assume that the statement holds for $n = k$, where k is an arbitrary positive integer. In other words, we assume that if p and q are positive integers such that $\max(p, q) = k$, then $p = q$, for some positive integer k.

Induction Step: Consider the case where $n = k + 1$, and let p' and q' denote two positive integers such that $\max(p', q') = k+1$. Then we must have $\max(p' - 1, q' - 1) = k$, which, by our induction hypothesis, implies that $p' - 1 = q' - 1$, and hence that $p' = q'$. This proves our claim.

2. [10 pts] Factor Fiction

Andrew loves playing a variation of BS with his friends where instead of BSing how many cards they have, they instead BS about the mathematical properties of the sum of their cards. Andrew is playing a game with $n + 1$ players (including him) for $n \in \mathbb{Z}^+$. The sum of each person's cards is a distinct integer ranging between 1 to $2n$. After the $n+1$ players randomly draw their hands, someone claims that for all pairs of players, their sums share common factors greater than 1. Prove that Andrew should call BS. (Prove there will always exist two players whose sums are relatively prime.)

3. [10 pts] Poker Proof

Daniel has placed first in the freshman poker tournament! Unfortunately, Kevin does not think this is possible, so in order to confirm that Daniel is worthy of first place, he challenges Daniel to prove that for all $x \in \mathbb{R}$, if $x + \frac{1}{x} \in \mathbb{Z}$, then $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

Help Daniel prove that he is worthy!