CIS 1600 Assigned: September 25, 2024

# Homework 5H

Due: 11:59PM EDT, September 30, 2024

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This homework is due electronically on Gradescope at 11:59PM EDT, September 30, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

**A. LaTeX**: All solutions are required to be typeset in LaTeX.

**B. Standard Deductions:**
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

**C. Solutions:** Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

**D. Collaboration:** You may not collaborate with anyone via any means.

**E. Citations:** All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

**F. Outside Resources:** Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

**G. Late Policy:** We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

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**1. [12 pts] Race to (Shoot) the Moon**

Yinqi is making it her life's goal to finally shoot the moon in Hearts. Luckily, she happens to know master Hearts player Olivia Hu-earts. Olivia agrees to train Yinqi in the moon shooting ways if she successfully solves an induction problem. Help Yinqi solve this problem so she can finally fulfill her dream.

Prove using induction that for all positive integers $n$, and for any integers $a$ and $b$ with $a \neq b$, $a^n - b^n$ is divisible by $a - b$.

**2. [16 pts] Bridging a Path Through the House of Cards**

After getting obsessed with mastering Bridge, Alex Oh, the "Ace Oh Spades," decides to host an epic Bridge tournament. However, being a competitive 1600 TA, she adds a twist to the tournament by designing a special Bridge lounge where players must navigate through walkways between different tables, each with an individual Bridge game taking place, during the tournament.

Alex's walkways connect $n \geq 2$ tables. For any two tables $S$ and $T$, there exists a walkway between $S$ and $T$. To make the tournament more interesting, they design each walkway to either go only from $S$ to $T$, only from $T$ to $S$, or in both directions.

Unfortunately, while Alex excels at playing Bridge, her interior design skills are not quite as strong. There's a problem: some of the tables might be dead-ends! This means that once players sit down at a dead-end table, there are no walkways to leave, which prevents them from continuing the tournament.

Maggie, the "Queen of Du-amonds," decides to join the tournament, but not before she figures out if there are any dead-end tables she should avoid. She is allowed to ask at most $2(n-1)$ questions to Alex, which take the form of "Can I get from table $S$ to table $T$ using a single walkway?" Each time, Alex will truthfully answer either "Yes" or "No". Prove, using induction, that if there are $n$ tables, Maggie can find a dead-end table, if it exists.

**3. [14 pts] Go Recursion?**

The CIS 1600 TAs are playing Go Fish, a highly intellectual and skilled card game! William wants to join in the fun as well. However, some of the other TAs, like the known gatekeeper Patrick, doubt his abilities to Go Fish adequately. They will only let him play if he can demonstrate his skill with recurrence relations and induction! Help William solve these problems to show that he meets the minimum intelligence requirements to play Go Fish!

(a) Consider the following sequence,

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

Determine a recurrence relation $S(n)$ that captures this sequence, where $n \geq 0$. Make sure to specify the base cases.

(b) Prove **using induction** that

$$S(n) = \varphi^n + \psi^n, \qquad \varphi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}$$

is another way of directly specifying $S(n)$. *Hint: what are some algebraic relationships between $\varphi$ and $\psi$?*

Please note that $\varphi$ is `\varphi` and $\psi$ is `\psi` in LaTeX.

**4. [10 pts] Some Grace-full Gridlock**

Grace (S) Liu and Grace (Y) Liu are setting up 1 vs. 1 solitaire in a 2 meter × 3 meter grid. However, not wanting to see beef between the Graces, Cindy knocks the small rectangular cards they have in their hands onto the ground. Specifically, 55 cards were knocked out of their hands INTO the play area (where the card is "in" the play area if the center of the card is in the play area). Show that when measured center-to-center, at least two of the cards will be within at most $\frac{1}{\sqrt{3}}$ meters of each other.

**5. [18 pts] Flip**

One quiet day in a casino, two mysterious figures walk in and sit down at a table dealing a new experimental magic card game called "Flip".

In the game of "Flip," there are $n$ total hearts (H) and spades (S) cards placed face up in a line. One can choose any face up hearts card and "flip" it face down (D). When such an event happens, the adjacent cards to the left and right will magically swap suits (hearts become spades and spades become hearts). One wins if they can successfully flip every card face down. Conversely, one loses if they have no remaining hearts to flip and not all cards are face down. Note that the only operation allowed is to flip hearts cards. (Once a card is flipped face down it will never be face up again).

This game has been raking in cash all night and the dealer, the Dilini-er as some call her, sees these newcomers as just another victim of "Flip". Little does she know, however, that these figures are Rajiv, aka the Master of Chance, and Eric, aka Errorless-Eric, both of whom are skilled in the arcane art of discrete math. Of course, upon seeing this game, they think: "Ahh, a great problem to test the power of discrete math." Yet they think this game is something they want the CIS 1600 students to think about.

Prove that there is a winning strategy in a game of flip on $n$ cards if and only if there are and odd number of hearts cards initially.

As Rajiv does in lecture, he gives you, his student, an example of a game of Flip. The game starts with the following configuration

"S", "H", "H", "S"

Rajiv then flips the second card from the left, the configuration becomes

"H", "D", "S", "S"

now Rajiv flips the only card that is Hearts and the configuration becomes

"D", "D", "S", "S"

As there are no hearts cards left and not all cards are face down, Rajiv and Eric realize that, for the first time ever, they have lost.