CIS 1600

Homework 4H

Assigned: September 18, 2024

Due: 11:59PM EDT, September 23, 2024

This homework is due electronically on Gradescope at 11:59PM EDT, September 23, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. LaTeX: All solutions are required to be typeset in LaTeX.

B. Standard Deductions:
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: You may not collaborate with anyone via any means.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

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1. **[8 pts] Who's New, Scooby-Doo?**

   For new TA initiation, Andrew decides to host a Scooby-Doo watch party for all $n \in \mathbb{Z}^+$ episodes. To avoid overflowing his 2nd floor Rodin dorm, he only invites new TAs to attend the watch party. Every new TA will watch at least one episode. For each episode $i \in \{1, 2, \dots, n\}$, Andrew then defines a (possibly empty) set S_i containing all new TAs who watched that episode. Prove by induction that the total number of TAs attending the Scooby-Doo watch party is at most the sum of $|S_i|$ for all $i \in \{1, 2, \dots, n\}$.

   Please make sure to first phrase the claim using mathematical notation, and only proofs (using mathematical notation) by induction will receive credit.

2. **[10 pts] A.L.T.A. (At Least Three Alex-tars)**

   Water. Earth. Fire. Air. Long ago, the four nations lived together in harmony. Then, everything changed when the Fire Nation attacked. Only the Alex-tars, masters of all four elements, could stop them, but when the world needed them most, they vanished. A hundred years pass and you discover a group of 160 benders who can each bend at least one of the four elements: 124 can bend water, 113 can bend earth, 127 can bend fire, and 119 can bend air. Prove that there exist at least three Alex-tars in the group of 160 benders to restore peace and unity to the world!

3. **[10 pts] Sid's Science Simulation**

   Sid(hant) the Science Kid is trying to find the optimal integer amounts of different materials to include in his science experiment. He decides to use amounts equal to the coefficients in the binomial expansion $(5x + 5)^k$ for some arbitrary $k \in \mathbb{Z}$, $k \geq 2$. However, for some reason, his scale will always output a value in the range $\left[0, \lfloor \frac{k}{2} \rfloor\right)$; if he attempts to weigh any object above this range, his scale will output the remainder of that weight when divided by $\lfloor \frac{k}{2} \rfloor$.

   Sid is convinced this will still let him uniquely identify his ingredient weights. However, as a skilled 1600 student, you know this isn't the case. Prove that there exists at least two coefficients with distinct numeric values for which the scale will report the same weight.

   Hint: If it helps, you may assume that $\binom{n}{m} \neq \binom{n}{l} \quad \forall\, l$ such that $1 \leq l \leq n \wedge l \neq m, (n - m)$

4. **[8 pts] They're $\frac{1}{2}$, they're $\frac{1}{4}$, they're $\frac{1}{6}$, they're $\frac{1}{8}$**

   Thomas (the Tank Engine) has made a fascinating discovery: the sum of the reciprocals of the natural numbers is unbounded! He excitedly tells his train friends, who share his enthusiasm for this proposition. However, Daniel "Diesel" Li is skeptical, and demands proof. Help Thomas and his friends by showing via induction that $\forall n \in \mathbb{Z}^+$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \ln(n + 1).$$

Hint: You may find it useful that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

5. [10 pts] Friendship is Magic

Vincent is deciding if he wants to watch Regular Show or My Little Pony. After much deliberation, he settles on My Little Pony because of Twilight Sparkle's uncanny ability to teach lessons about friendship. All Vincent wants to do is watch his show, but he realizes he still needs to bypass the parental lock on his TV which will only open with the proof to the following identity: For all integers $n \geq 4$,

$$\binom{\binom{n}{2}}{2} = 3\binom{n}{3} + 3\binom{n}{4}$$

Help Vincent fulfill his only wish by giving him a combinatorial proof to the claim!

6. [12 pts] "Everything Is Not What It Seems"

Back in the Russo family basement, Jerry is highly suspicious that Alex (played by Selena Gomez) has not done her math homework properly! And he's right, because Alex was secretly using magic to help write her proofs. Now, Jerry is threatening to ground Alex for 2 weeks if she cannot prove that she didn't use magic to do her homework. The irresponsible teen she is, she's passed the problem along to you, and threatens to curse you with her silly goose spell! Help save Alex (and yourself) by proving the following claim!

Prove via induction that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all $n \in \mathbb{Z}^+$.

Hint: If you find yourself stuck in this proof, consider proving a stronger claim.

7. [12 pts] Ferb, I know what we're going to do today

During one of their 104 days of summer vacation, Phineas and Ferb create their greatest invention yet: a map of DRL's basement, and want to give it to a deserving Penn student. To find out who the most deserving is, they come up with a game to find their "Chosen One". There will be $n \geq 1$ students, and each student will go up against every other student in a one-on-one math competition with one clear winner and one clear loser (no draws can occur!).

Note that victories are not transitive—student a beating student b and student b beating student c does not necessarily imply that a will have beaten c, as well.

Phineas and Ferb set the following criteria for an student to be the Chosen One: a student x is the Chosen One if for all other students y, either x beats y or x beats some third student z who beat y.

Help Phineas and Ferb determine the most deserving student and prove that at least one of the n students will be the Chosen One.