CIS 1600

Homework 15H

Assigned: December 4, 2024

Due: 11:59PM EDT, December 9, 2024

This OPTIONAL homework is due electronically on Gradescope at 11:59PM EDT, December 9, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. LaTeX: All solutions are required to be typeset in LaTeX.

B. Standard Deductions:
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: Please make sure to strictly follow our collaboration policy as clarified on Ed.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

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1. [18 pts] Bike or Bust

This Christmas, Alex Y. and Alex O. want a bicycle with their name engraved on it. Unfortunately, Santa has an elf shortage this winter and was only able to create one "Alex" bicycle. To give the bicycle to the most deserving person, Santa comes up with a statement regarding Ramsey Numbers for both Alex's to prove: $R(k,l) \leq R(k-1,l)+R(k,l-1)$ for $k > 2$ and $l > 2$. Help the Alex of your choice prove this conjecture so they can bicycle away into the horizon!

2. [18 pts] Color Me Intrigued

In preparation for Christmas, David is adorning his Christmas Tree with a set S of ornaments. Each of the ornaments are a certain color, and in total there are k distinct colors. Staring at his set S of ornaments, David discovers an ingenious idea: He can define a coloring on the elements in a set, similar to how his CIS 1600 students define a coloring on the vertices in a graph! David defines the following:

- A k-coloring of a set A is an assignment of colors to the elements in the set A, where each element in A is assigned one color drawn from a set of k colors.
- Unlike a graph, where neighboring vertices must have distinct colors, there is no relationship between the colors of different elements in the set.

Help David prove the following ingenious properties of set colorings!

Let $S_1, S_2, \ldots, S_n$ be arbitrary subsets of the set S, where $\forall i, 1 \leq i \leq n, |S_i| = \ell$.

(a) Prove that if $n < 3^{\ell-1}$, then there exists a 3-coloring of elements in S such that no S_i is monochromatic.

(b) Prove that if $n < \frac{3^{(\ell-1)}}{2^\ell}$, then there exists a 3-coloring of elements in S such that, for all i, S_i contains at least one element of each of the three colors.

3. [18 pts] Hanukkah Hullabaloo

Max is preparing for Hanukkah, but he knows seven of his eight menorah candles do not light and the working candle always lights. To find the working candle, he places the eight candles in the menorah. He then chooses a candle uniformly at random from the menorah and attempts to light it using the shamash candle (the ninth candle). If the selected candle does not light, he replaces it and repeats. Max does this until the candle lights *or* until he has tested n candles. Note that the shamash candle is not included in the eight candles being tested.

Let X denote the number of candles Max tries to light. *You do not need to define a sample space. Additionally, you do not need to give your expression for variance in closed form for this question.*

(a) Find $\mathbb{E}[X]$.

(b) Find $\mathrm{Var}[X]$.

4. [16 pts] A New Tradition

This year, Sid plans on going to Times Square to watch the New Year's Ball Drop live! From where he parked his car, there are n train stops to get to Times Square. For each of the n stops, the stop can be red or green. For any two routes a, b of length n (the number of stops in each route is n), let the distance between a and b denoted as $\Delta(a, b)$ be the number of stops on which the colors differ (Ex. if $n = 4, a = RGGR, b = RGRG$, the distance between a and b is 2).

Help Sid reach the event on time by proving the following claim:

> For a sufficiently large $n \geq 6$, show there exists a set S of $\sqrt{n}/2$ possible routes of length n such that for any pair of routes $x, y \in S$, $\Delta(x, y) > n/3$.