CIS 1600

Homework 11H

Assigned: November 6, 2024

Due: 11:59PM EDT, November 11, 2024

This homework is due electronically on Gradescope at 11:59PM EDT, November 11, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. LaTeX: All solutions are required to be typeset in LaTeX.

B. Standard Deductions:
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: Please make sure to strictly follow our collaboration policy as clarified on Piazza.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

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1. [18 pts] The Search for Bigfoot

Eric is forming a search team for his annual Bigfoot hunt. He polls n distinct TAs on CIS 1600 staff to gauge interest. The probability a TA joins Eric on his expedition is p and the probability they stay back is $1-p$, independent from any other TA's decision. Let A be the total number of TAs (not including Eric himself) that join Eric and let B be the number that don't join him.

(a) Determine the probability mass function for random variable A.

(b) Determine the expected value of A.

(c) Determine the variance of A.

(d) What is the probability that the first TA to be polled ends up being the only person to make their decision?

(e) What is the probability that A or B equals 1?

(f) What is the expectation and the variance of the difference $D = A - B$?

2. [14 pts] CIS 1600 is FINtastic!

Our favorite mermaids on staff, Maggie and Megan, have been trying to visit every restaurant in Atlantis! Atlantis is made up of restaurants and canals. Each canal runs bidirectionally between exactly two restaurants, with each pair of restaurants having at most 1 canal between them.

For the arrangement of restaurants and canals $G = (R, C)$, the mermaids define the following types of subsets

- *Independent Set* – A set of restaurants $S \subseteq R$, such that for any two distinct restaurants $u, v \in S$, $\{u, v\} \notin C$.
- *Clique* – A set of restaurants $S \subseteq R$, such that for any two distinct restaurants $u, v \in S$, $\{u, v\} \in C$.

Further, they denote:

- $\alpha(G)$ as the size of the maximum independent set in G
- $\kappa(G)$ as the size of the maximum clique in G
- $\chi(G)$ as the minimum number of distinct menus that would need to exist in G such that any two restaurants that share a canal have different menus. (Two restaurants that are directly joined by a single canal must not have the same menu)

Maggie and Megan are pretty sure there are some relationships between the symbols they came up with, but they're having trouble figuring out what they are specifically. For any arrangement of restaurants and canals, help the mermaids find the **strictest** relationship that holds between the following quantities (less than, less than or equal to, equal to, greater than or equal to, greater than, not related). Remember to prove your answers.

(a) $\alpha(G)$ and $\kappa(\overline{G})$.

(b) $\chi(G)$ and $\kappa(G)$.

3. [12 pts] Luke's Markovian Missteps

When Luke Tong-stellan isn't being a menace to society and betraying (spoiler for the Percy Jackson series!) all the other 1600 TAs, he enjoys proving facts about Markov's inequality. However, he was tripped up by the following proofs and needs you to help him!

(a) Show that Markov's inequality only applies to non-negative random variables. In other words, give an example of a random variable and its probability distribution for which Markov's inequality gives an incorrect answer.

(b) Suppose Z is a random variable that is always at least -8 and has expectation 0. Since Z can take on negative values, Markov's inequality does not apply directly. Still, show that the probability that $Z \geq 16$ is at most $\frac{1}{3}$. Use only Markov's inequality, and known properties about expectations and random variables.

4. [12 pts] Journey to the West

The legendary monkey Kevin Song-WuKong is on a journey to the west. In order to guide him, Luke Tong-Seng has given him a map with towns and bidirectional roads between the towns, with each pair of towns having at most 1 road between them. Along each road contains a hidden artifact, but the spirits guarding each town forbid them from visiting a town more than once. Luckily, Kevin Song-WuKong has the ability to fly to any town at will, though he can only explore a road by traveling between (and hence visiting) the towns incident on that road.

To strategize, Victor Zhu-BaJie and Ishaan Shah-HeShang each create a unique set of roads they can explore such that no additional roads could be added without visiting a town twice (i.e. the sets are *maximal*). Denote these sets of roads M_1 and M_2. Note that M_1 and M_2 are distinct, but may share some of the roads. Show that $|M_1| \geq \frac{1}{2}|M_2|$ to help the legendary team complete their journey to the west!

5. [14 pts] Pandora's Probability Puzzle

Pandora's box initially contains 10 "evils" and 0 "goods." She accidentally leaves the lid slightly open, allowing one randomly selected item to escape from the box at the end of each century, where each "good"/"evil" escapes with equal probability. Whenever this occurs, Prometheus will add a "good" into the box to give hope to humanity. After a millennium (10 centuries), a total of 10 items have escaped and 10 items remain in Pandora's box.

Let X be a random variable denoting the number of "evils" remaining in Pandora's box after a millennium. Find $\mathbb{E}[X]$ and $\mathrm{Var}[X]$. *You do not need to define a sample space for this problem.*