

Homework 10H

Due: 11:59PM EDT, November 4, 2024

This homework is due electronically on Gradescope at 11:59PM EDT, November 4, 2024. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. \LaTeX : All solutions are required to be typeset in \LaTeX .

B. Standard Deductions:

- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: Please make sure to strictly follow our collaboration policy as clarified on Piazza.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [10 pts] The Greenberg Primary

The CIS 1600 TAs are voting to see who will become the new Supreme Ruler of Greenberg Lounge! Instead of a debate, the two frontrunners Maggie and Jan will compete to demonstrate their discrete mathematics prowess to the rest of the staff. As her trusty student turned campaign intern, you're helping Maggie prepare to face Jan. She's confused about the following conjecture about graphs, and assigns you to prove or disprove it.

Let G be a simple, undirected graph with at least one edge. Construct another graph G' as follows — for each edge e in G , we create a unique corresponding vertex v_e in G' . Then, for any two vertices v_{e_1} and v_{e_2} in G' , there is a corresponding edge $\{v_{e_1}, v_{e_2}\}$ in G' if the edges e_1 and e_2 in G are incident on the same vertex. If G has an Eulerian circuit, then G' has a Hamiltonian cycle.

Help lead Maggie to victory by proving or disproving the statement!

2. [10 pts] Election Gambling

Election-beth is playing a betting game with her friends on who will win the 2044 election. She is betting on her favorite candidate, Suzanna Wang-shington.

For each of the n regions in the country, let Y_i ($1 \leq i \leq n$) be a random variable denoting the number of votes Suzzy gets from the i th region. After painstaking thought, Election-beth is confident that for each pair of regions Y_i, Y_j ($1 \leq i, j \leq n, i \neq j$) that

$$\mathbb{E}[Y_i Y_j] = \mathbb{E}[Y_i] \mathbb{E}[Y_j]$$

Election-beth wants to understand how safe her bet is by calculating $Var[Y]$ where Y denotes the random variable for the total number of votes Suzzy gets.

Help her show that when the events are pairwise independent, linearity of variance holds. That is, prove that

$$Var[Y] = \sum_{i=1}^n Var[Y_i]$$

3. [10 pts] Ishaan's ID

As an intern for the Federal Election Commission and first time voter, Ishaan has been given the opportunity to create his own 12-character voter ID. He is allowed to use uppercase letters, lowercase letters, and digits 0-9. Ishaan decides to generate his ID by selecting each character independently and uniformly at random.

His boss, a number lover, begins to wonder about the digits in the ID that Ishaan generates, and tasks Ishaan with answering the following questions:

- (a) What is the probability that exactly 6 of the characters are digits?

- (b) What is the expected number of digits in the ID?
- (c) What is the variance of the number of digits in the ID?

Help Ishaan answer these questions so he can get a return offer!

4. [10 pts] Big Gandhi is Watching You

The Gandhi Party is preparing for its “election” for the 4 members that represent the will of Big Gandhi in Oceania. Winston, sick of how Andrew, Cindy, Eric, and Dilini have always been representing Big Gandhi, decided to secretly raise support for the Zhang Party and the Pei Party. With his instigation, districts are now affiliated with one of the three parties: the Gandhi, Zhang, and Pei parties. Note that each district has a bidirectional edge between them if they border each other. These differences are represented in the graph by different colorings assigned to each district. Therefore, we can say that this graph of districts G is a connected graph with 3 or more vertices, and that $\chi(G) = 3$.

Winston observes that each district borders only districts of differing party affiliations (ignore how this would work geographically). Prove that there exists a Gandhi district that has both a Zhang district neighbor and a Pei district neighbor.

5. [10 pts] Lottocracy

Good news—Winston has successfully toppled the Gandhi regime! However, in the rush to fill the power vacuum, Oceania’s new election now has 30 distinct political candidates. The once-united cause for liberty has since fractured, with all voters now supporting either the Zhang Party or the Pei Party (but not both).

Luke is encouraging everyone to go to the polls, and he witnesses 30 Zhang-affiliated voters and 30 Pei-affiliated voters each cast a vote for one of the 30 candidates on the ballot. Overwhelmed by choices, each of the 60 voters independently and uniformly at random picks one of the 30 candidates to vote for, completely ignoring their party affiliation/political views.

To maintain a bipartisan and diverse viewpoint, Luke hopes to see a good mix of votes from both parties for each candidate. Help Luke by calculating the expected number of candidates who received votes from at least one voter from each party.

6. [10 pts] Darren’s Dilemma

The day before the election, Darren comes up with the great idea of committing voter fraud in the upcoming election by submitting n ballots, one of them real and $n - 1$ of them fake.

Right before he votes, he has a change of heart, but the ballots all look alike, so he can’t tell which one is real. To determine the real one, he tries to insert them one by one into the voting machine, which rejects all the fake ballots and will always accept the real ballot. After it accepts

the real ballot, it will prevent him from inserting any more.

Let X be a random variable denoting the number of ballots Darren inserts into the voting machine until the voting machine accepts no more ballots. Find $\text{Var}[X]$.

You may assume that $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

7. [10 pts] Discreteopia's Disparity

The prosperous nation of Discreteopia is ruled by the benevolent dictator Rajiv. Despite his absolute control, the nation naturally still holds sham elections to give its citizens the illusion of choice. Running against Rajiv is his right-hand woman, Cindy. In initial election polls, $N \geq 160$ people respond and exactly $3N/5$ people are in favor of Rajiv.

- (a) You sample 160 people with replacement from those polled, uniformly at random. Let X be the **proportion** in favor of Rajiv (i.e. if 80 of the people in the sample are in favor of Rajiv, $X = 1/2$). Find $\text{Var}(X)$.

Hint: You may find linearity of variance useful for this problem as claimed in Question 2.

- (b) Prove the proportion of people in favor of A in our sample won't be too large by showing $\Pr(X > \frac{3}{4}) < \frac{1}{15}$.