Solutions to Practice Problems

P1: You have a biased coin, which shows heads with probability $0 < p < 1$ and tails with probability $1 - p$.

(a) How can you simulate a fair coin? The only requirement is that the expected number of times you flip the coin is finite.

(b) In expectation, how many coin tosses are needed to simulate one fair toss?

Solution:

(a) Look at the results after two consecutive flips of this coin: HT, TH, HH, TT. It is clear that HT and TH are equally likely (both have probability $p \cdot (1-p)$). So, what one can do to simulate a fair coin is the following. Flip this biased coin two times. If the result is one of HT, TH, then assume HT is heads and TH is tails. If the result is something else, repeat.

(b) Let us make sure we expect a finite number of flips. Call two flips of the biased coin a trial and achieving HT or TH a success. Then the expected number of trials to reach a success comes from a geometric distribution with probability of success $2 \cdot p \cdot (1-p)$. So, the expected number of trials is $\frac{1}{2 \cdot p \cdot (1-p)}$, and since each trial is two tosses, the expected number of tosses needed to simulate a fair toss is $2 \cdot \frac{1}{2 \cdot p \cdot (1-p)} = \frac{1}{p \cdot (1-p)}$.

P2: You want to determine your lucky number. To do this, you flip 100 fair coins. You total the number of heads, H, and number of tails, T. You denote your lucky number to be $H - T$. What is the expected value and variance of your lucky number?

Solution:

H and T are binomial random variables, so we know $E[H] = 0.5 \times 100 = 50 = E[T]$. In addition, $Var[H] = Var[T] = 100 \cdot 0.5 \cdot 0.5 = 25$. Note, by linearity of expectation, $E[H - T] = 0$, which is the expected value of your lucky number. Also, it is helpful to observe that $T = 100 - H$, so your lucky number can also be expressed as $H - T = H - (100 - H) = 2H - 100$. Hence,

$$
\begin{aligned}
Var[H - T] =& E[(H-T)^2] - 0^2 \\
=& E[(2H - 100)^2] \\
=& E[4H^2 - 400H + 10000] \\
=& 4 \cdot E[H^2] - 400 \cdot E[H] + 10000
\end{aligned}
$$

What is $E[H^2]$? Well, rearranging the definition of variance,

$$E[H^2] = E[H]^2 + Var[H]$$
$$E[H^2] = 50^2 + 25 = 2525$$

Now, we can finally go back to finding the variance of our lucky number,

$$Var[H - T] = 4 \cdot 2525 - 400 \cdot 50 + 10000$$
$$Var[H - T] = 100$$

P3: Let $G = \{V_1, V_2, E\}$ be a bi-partite graph. Further, G is Hamiltonian. What does this imply about the relative sizes of V_1 and V_2? Prove your claim.

Solution:

$|V1| = |V2|$.

Proof by contradiction: without loss of generality, assume $|V1| > |V2|$ and that G has a cycle which traverses all vertices. We will follow this cycle by starting at some arbitrary vertex $u \in V2$. After $2 \cdot |V2|$ edges have been traversed, we will have arrived again at u, which terminates the cycle but has left out $|V1| - |V2|$ vertices. Contradiction.

P4: Consider a cube, C, where every corner is a vertex. Does C have a perfect matching? Is C Hamiltonian?

Solution:

Yes and yes. An example of a perfect matching would be any four parallel edges. For the sake of giving a hamiltonian cycle, H, let us first place this cube in the first quadrant of the 3-D space, or $\mathbb{R}^3$. Then H is $(0,0,0) \to (1,0,0) \to (1,0,1) \to (0,0,1) \to (0,1,1) \to (1,1,1) \to (1,1,0) \to (0,1,0) \to (0,0,0)$.

P5: Let $f : X \to Y$ be some function. If $S \subseteq X$, then define $f(S) = \{f(s) \mid s \in S\}$. Let $A, B \subseteq X$. Prove that $A \subseteq B$ implies $f(A) \subseteq f(B)$.

Solution:

Assume $A \subseteq B$. We would like to show $f(A) \subseteq f(B)$. Take some arbitrary element $y \in f(A)$. Then, we know there exists some $x \in A$ such that $f(x) = y$. Since $A \subseteq B$, $x \in B$. It follows that $y \in f(B)$ by definition of $f(B)$, and we are done.

P6: Define a relation R on $\mathbb{N}$ where $(a, b) \in R$ if and only if a and b have no positive common factors other than 1. For each of the 5 properties of relations that we have studied, state and prove whether R has this property.

Solutions:

Reflexivity: R is NOT reflexive.

R would be reflexive iff $\forall x \in \mathbb{N}, (x, x) \in R$. However, consider $x = 2$. $(2, 2) \notin R$ since 2 and 2 DO have a positive common factor other than 1, namely 2. Thus, R is not reflexive.

Irreflexivity: R is NOT irreflexive.

R would be irreflexive iff $\forall x \in \mathbb{N}, (x, x) \notin R$. However, consider $x = 1$. $(1, 1) \in R$ because the only positive common factor between 1 and 1 is 1. Thus, R is not irreflexive.

Symmetry: R IS symmetric.

Let $x, y \in Ast.(x, y) \in R$. Then x and y have no positive common factors other than 1. This means that y and x have no positive common factors other than 1. It follows that $(y, x) \in R$. Therefore, $(x, y) \in R \implies (y, x) \in R$. Thus, R is symmetric.

Antisymmetry: R is NOT antisymmetric.

R would be antisymmetric iff $(x, y) \in R \wedge (y, x) \in R \implies x = y$. However, consider $x = 2$ and $y = 3$. Since 2 and 3 have no positive common factors other than 1, it must be that $(2, 3) \in R$. Likewise, since 3 and 2 have no positive common factors other than 1, it must be that $(3, 2) \in R$. However, $2 \neq 3$. Thus, R is not antisymmetric.

Transitivity: R is NOT transitive.

R would be transitive iff $(x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$. However, consider $x = 2$, $y = 3$, $z = 4$. Since 2 and 3 have no positive common factors other than 1, it must be that $(2, 3) \in R$. Since 3 and 4 have no positive common factors other than 1, it must be that $(3, 4) \in R$. However, $(2, 4) \notin R$ since 2 and 4 DO have a positive common factor other than 1, namely 2. Thus, R is not transitive.

P7: Let X be a geometric random variable with parameter p. What is $\mathrm{Var}[X]$?

Solution. By definition, $\mathrm{Var}[X] = \mathbf{E}[X^2] - \mathbf{E}[X]^2$. We know that $\mathbf{E}[X]^2 = 1/p^2$. Let Y be a random variable that is 1 if the first coin toss is a Heads, 0, otherwise.

$$\begin{aligned}
\mathbf{E}[X^2] &= \Pr[Y=1]\mathbf{E}[X^2 \mid Y=1] + \Pr[Y=0]\mathbf{E}[X^2 \mid Y=0] \\
&= p + (1-p)\mathbf{E}[(X+1)^2] \\
&= p + (1-p)\mathbf{E}[X^2] + 2(1-p)\mathbf{E}[X] + (1-p) \\
&= (1-p)\mathbf{E}[X^2] + \frac{2-2p}{p} + 1 \\
p\mathbf{E}[X^2] &= \frac{2-p}{p} \\
\mathbf{E}[X^2] &= \frac{2-p}{p^2}
\end{aligned}$$

Thus, we have

$$\mathrm{Var}[X] = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

P8: You go to a vending machine with 11 different candy bars, but you only like one type. The vending machine only has one candy bar left of each type, and since it is broken, it randomly releases a candy bar every time you pay (it always releases a candy bar). You will get the candy bar you like on your nth try.

(a) What is the expected value of n?

(b) Now let's say that the vending machine has an infinite supply of each of the candy bars. Do you think the expected value of n now is higher or lower than before? What is the new expected value?

Solution.

(a) We seek $\mathbf{E}[n]$:

$$\mathbf{E}[n] = \sum_{k=1}^{11} \Pr[n=k] \times k$$

Since any possible ordering that the vending machine dispenses the candy bars is equally likely, $\Pr[n=k] = \frac{1}{11}$ regardless of k (there are other ways to conclude that $\Pr[n=k] = \frac{1}{11}$). Hence:

$$\begin{aligned}
\mathbf{E}[n] &= \sum_{k=1}^{11} \Pr[n=k] \times k \\
&= \sum_{k=1}^{11} \frac{1}{11} \times k \\
&= \frac{1}{11} \times \sum_{k=1}^{11} k \\
&= \frac{1}{11} \times \frac{11 \times 12}{2} = 6
\end{aligned}$$

Since all outcomes are equally likely, it makes sense that you expect to get your candy bar right at the middle.

(b) If the vending machine has an infinite supply of each candy bar, the expected value should increase since now the vending machine can supply you with a candy bar you dislike multiple times. We can compute the new expected value similarly to part (a):

$$\mathbf{E}[n] = \sum_{k=1}^{11} \Pr[n = k] \times k$$

Note that $\Pr[n = k] = \left(\frac{10}{11}\right)^{k-1} \times \frac{1}{11}$:

$$\begin{aligned} &= \sum_{k=1}^{11} \left(\frac{10}{11}\right)^{k-1} \times \frac{1}{11} \times k \\ &= \frac{1}{11} \times \sum_{k=1}^{11} \left(\frac{10}{11}\right)^{k-1} \times k \end{aligned}$$

Now, we can solve this using algebra and geometric series formulas, and it would be right. But we can also notice a nice recursive property in the situation which makes the computation very easy. If you get your preferred candy bar on the first try, then it takes you exactly one try obviously. If you do not get your preferred candy bar on the first try, then you are in the same situation as originally! You expect to take E[n] more tries, in addition to the one you have already failed with. Thus we solve the relation:

$$\begin{aligned} \mathbf{E}[n] &= \frac{1}{11} \times 1 + \frac{10}{11} \times (\mathbf{E}[n] + 1) \\ &= 11 \end{aligned}$$