Mathematical Foundations of Computer Science
Practice Problems

Given: December 10, 2023

P1: You have a biased coin, which shows heads with probability $0 < p < 1$ and tails with probability $1 - p$.

(a) How can you simulate a fair coin? The only requirement is that the expected number of times you flip the coin is finite.

(b) In expectation, how many coin tosses are needed to simulate one fair toss?

P2: You want to determine your lucky number. To do this, you flip 100 fair coins. You total the number of heads, H, and number of tails, T. You denote your lucky number to be $H - T$. What is the expected value and variance of your lucky number?

P3: Let $G = \{V_1, V_2, E\}$ be a bi-partite graph. Further, G is Hamiltonian. What does this imply about the relative sizes of V_1 and V_2? Prove your claim.

P4: Consider a cube, C, where every corner is a vertex. Does C have a perfect matching? Is C Hamiltonian?

P5: Let $f : X \to Y$ be some function. If $S \subseteq X$, then define $f(S) = \{f(s) \mid s \in S\}$. Let $A, B \subseteq X$. Prove that $A \subseteq B$ implies $f(A) \subseteq f(B)$.

P6: Define a relation R on $\mathbb{N}$ where $(a, b) \in R$ if and only if a and b have no positive common factors other than 1. For each of the 5 properties of relations that we have studied, state and prove whether R has this property.

P7: Let X be a geometric random variable with parameter p. What is $\text{Var}[X]$?

P8: You go to a vending machine with 11 different candy bars, but you only like one type. The vending machine only has one candy bar left of each type, and since it is broken, it randomly releases a candy bar every time you pay (it always releases a candy bar). You will get the candy bar you like on your nth try.

(a) What is the expected value of n?

(b) Now let's say that the vending machine has an infinite supply of each of the candy bars. Do you think the expected value of n now is higher or lower than before? What is the new expected value?