CIS 160

Exam 2 Solutions

November 15, 2018

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1. AJ decides to choose one of three biased coins (coin A, coin B, coin C) and flip the chosen coin once. Coin A shows heads with probability $\frac{5}{15}$, coin B shows heads with probability $\frac{3}{15}$, and coin C shows heads with probability $\frac{1}{15}$. Suppose that AJ chooses coin A with probability $\frac{1}{4}$, coin B with probability $\frac{1}{4}$ and coin C with probability $\frac{1}{2}$. What is the probability that AJ chose coin C, given that the coin flip resulted in heads? You do not need to simplify your answer.

Solution. We consider the following events.

H: event that AJ's coin flip results in heads.
A: event that AJ picked coin A.
B: event that AJ picked coin B.
C: event that AJ picked coin C.

We want to find $\Pr[C \mid H]$.

$$\begin{aligned}
\Pr[C \mid H] &= \frac{\Pr[C \cap H]}{\Pr[H]} \\
&= \frac{\Pr[C]\Pr[H \mid C]}{\Pr[A \cap H] + \Pr[B \cap H] + \Pr[C \cap H]} \\
&= \frac{(1/2)(1/15)}{\Pr[A]\Pr[H \mid A] + \Pr[B]\Pr[H \mid B] + \Pr[C]\Pr[H \mid C]} \\
&= \frac{1/30}{(1/4)(5/15) + (1/4)(3/15) + (1/2)(1/15)} \\
&= \frac{1/2}{5/2} \\
&= \frac{1}{5}
\end{aligned}$$

2. Answer the following questions. No justification is required. No partial credit will be given for incorrect answers.

a. **Find the flaw or state there is none.**
Claim: For every non-negative integer n, $5n = 0$.

Base Case: $n = 0$. We see $5(0) = 0$ and our claim holds in this case.

Induction Hypothesis: Assume that for some integer k, $5j = 0$ for every integer j, $0 \leq j \leq k$.

Induction Step: Now, we want to prove that $5(k+1) = 0$. First, we write $k+1 = i+j$ where i and j are non-negative integers less than $k+1$. We then note:

$$\begin{aligned} 5(k+1) &= 5(i+j) \\ &= 5i + 5j \\ &= 0 + 0 && \text{(By the IH)} \\ &= 0 \end{aligned}$$

Thus our claim is proven by induction.

Solution. The flaw is when $k = 0$. We see there is no way to write $k+1 = 1$ as the sum of two non-negative integers less than 1.

b. I shuffle a standard deck of 52 cards thoroughly so that every possible ordering is equally likely. Let E denote the event that the top card is a spade. Let F denote the event that the 3rd card from the top is a spade. We can show that $\Pr[E] = 1/4$. Is $\Pr[F]$ larger than, equal to, or smaller than $1/4$?

Solution. $\Pr[F] = 1/4$. The location does not matter.

c. Let $\Omega = \{a, b, c, d, e\}$ be a uniform sample space. Let A be the event given by $\{a, b, c\}$. Give an example of an event B such that $\Pr[B] > 0$ and A and B are independent.

Solution. $B = \Omega$.

d. Let $\Omega = \{a, b, c, d\}$ be a uniform sample space. Produce three events A, B, and C that are pairwise independent, but not mutually independent.

Solution. Let $A = \{a, b\}, B = \{b, c\}$, and $C = \{b, d\}$.

3. An edge $\{u, v\}$ is a *chord* of the cycle C in an undirected simple graph $G = (V, E)$ if u and v are vertices in the cycle, but the edge $\{u, v\} \in E$ is not an edge of the cycle.

Prove the following statement: If $G = (V, E)$ is a simple undirected graph with minimum degree at least 3 then G contains a cycle with a chord.

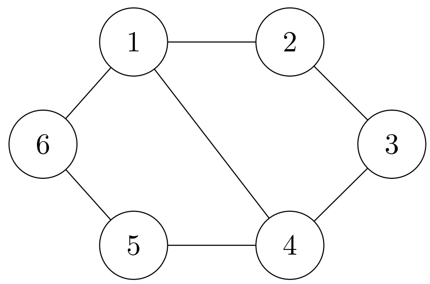


Figure 1: An example of a cycle with chord $\{1, 4\}$.

Solution. Let $P : v_1, v_2, \ldots, v_k$ be a maximal path in G. Since $\deg(v_k) \geq 3$ and since P is maximal, for some p and q, where $p < q < k - 1$, there must be vertices v_p and v_q in P such that $\{v_k, v_p\} \in E$ and $\{v_k, v_q\} \in E$. Thus, the path $v_p, v_{p+1}, \ldots, v_k$ combined with the edge $\{v_k, v_p\}$ forms a cycle and the edge $\{v_k, v_q\}$ forms a chord in the cycle.

4. Suppose that we flip a fair coin until either it comes up tails twice (not necessarily consecutively) or we have flipped it six times. What is the expected number of times we flip the coin? You do not need to simplify your answer.

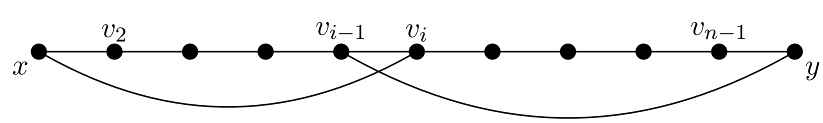
Solution. Let F be the random variable denoting the number of flips. We want to find $\mathbf{E}[F]$. Note that the probability of getting any given series of flips of length x is $(\frac{1}{2})^x$ and that we stop on the sixth flip regardless of the outcome. A series which ends after $f = 2, 3, 4,$ or 5 flips must end in a tails and have exactly one tails in the first $f - 1$ flips (so there are $f - 1$ ways this could happen). A series which ends after $f = 6$ flips must have at most one tails in the first five flips (there are 6 ways this can happen: one with zero tails, one for each spot the one tails could go in). Then, the probability distribution of F is given by

$$\Pr[F = f] = \begin{cases} (f-1)\left(\frac{1}{2}\right)^f & f = 2, 3, 4, 5 \\ 6 \cdot \frac{1}{2^5} & f = 6 \end{cases}$$

$$\begin{aligned} \mathbf{E}[F] &= \sum_{f=2}^{6} f \Pr[F = f] \\ &= 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{2}{2^3} + 4 \cdot \frac{3}{2^4} + 5 \cdot \frac{4}{2^5} + 6 \cdot \frac{6}{2^5} \\ &= \frac{15}{4} \end{aligned}$$

5. For any integer $n \geq 3$, let G be a simple undirected graph on n vertices such that for any two vertices u and v in G, it is true that $\deg(u) + \deg(v) \geq n$. Prove that G has a Hamiltonian cycle. For this question only, you may not refer to any lemmas from lecture, recitation or homeworks.

Solution. Assume for contradiction that G does not have a Hamiltonian cycle. Add new edges to G one-by-one, until we come to a point where adding an edge, say (x, y), creates a Hamiltonian cycle. Let G' be the graph in which all vertices have degree such that the sum of any two degrees is at least n and G' does not have a Hamiltonian cycle, but adding (x, y) will make G' Hamiltonian. Since adding edge (x, y) creates a Hamiltonian cycle in G', it must be that G' has a Hamiltonian path that begins at x and ends at y. Let the path be $x = v_1, v_2, \ldots, v_{n-1}, v_n = y$. We now apply the pigeon-hole principle as follows. Let the pigeons be the edges incident on the vertices x and y (at least n in total), and let the holes be the $(n-1)$ edges of the form (v_i, v_{i+1}), where $1 \leq i \leq n-1$. An edge (pigeon) of the form (x, v_i) is assigned to the "hole" (v_{i-1}, v_i) and an edge (pigeon) of the form (y, v_i) is assigned to the "hole" (v_i, v_{i+1}). Since $\deg(x) + \deg(y) \geq n$ and at most one edge incident on x (or y) is assigned to a hole, by the pigeon-hole principle, there must be i such that $3 \leq i \leq n-1$ and there is an edge (x, v_i) and an edge (y, v_{i-1}) (see figure below). Note that since the edge (x, y) does not exist in G', the hole corresponding to (v_1, v_2) only has one edge, namely (x, v_2). Similarly, the hole (v_{n-1}, v_n) will only contain the edge (y, v_{n-1}). But this would mean that $xv_2v_3 \cdots v_{i-1}yv_{n-1}v_{n-2} \cdots v_i$ is a Hamiltonian cycle, a contradiction.



6. There are n companies and n applicants. Each company has a preference list that ranks every applicant and each applicant has a preference list that ranks each company. There are no ties in the preference lists. We say that a company c and an applicant a form a matching pair if the applicant a is the highest ranked applicant (most preferred applicant) on c's list and if the company c is the highest ranked company (most preferred company) on a's list. Assuming that the preference lists of every company and every applicant are independently and uniformly generated over all permutations of n applicants and over all permutations of n companies, respectively, what is the expected number of matching pairs?

Example. Suppose we have three applicants a_1, a_2, a_3, having the preference lists (from highest ranked to lowest):

a_1: (c_1, c_2, c_3)
a_2: (c_2, c_1, c_3)
a_3: (c_3, c_1, c_2)

and three companies c_1, c_2, c_3, having the preference lists:

c_1: (a_1, a_3, a_2)
c_2: (a_2, a_1, a_3)
c_3: (a_1, a_3, a_2)

Then, we would get two matching pairs, (a_1, c_1) and (a_2, c_2).

Solution. We define the following random variables.

X: random variable that denotes the number of matching pairs.
X_i: indicator random variable that is 1 if and only if applicant i is part of a matching pair.

Note that applicant i is part of a matching pair if and only if i's most preferred company also ranked i first. Additionally, the probability of getting any arrangement of rankings is equal and so our sample space is uniform. Therefore, $\Pr[X_i = 1] = \dfrac{1 \times (n-1)! \times (n!)^{2n-1}}{(n!)^{2n}} = \dfrac{1}{n}$. Thus we have u

$$
\begin{aligned}
X &= \sum_{i=1}^{n} X_i \\
\mathbf{E}[X] &= \sum_{i=1}^{n} \mathbf{E}[X_i] && \text{(Linearity of Expectation)} \\
&= \sum_{i=1}^{n} \Pr[X_i = 1] \\
&= \sum_{i=1}^{n} \frac{1}{n} \\
&= 1
\end{aligned}
$$

Alternate Solution. We define the following random variables.

X: random variable that denotes the number of matching pairs.
X_i: indicator random variable that is 1 if and only if the i^{th} pair is matching.

Note that the i^{th} pair (a, c) is matching if and only if a and c both ranked each other first. Additionally, the probability of getting any arrangement of rankings is equal and so our sample space is uniform. Then $\Pr[X_i = 1] = \frac{1 \times (n-1)! \times 1 \times (n-1)! \times (n!)^{2n-2}}{(n!)^{2n}} = \frac{1}{n^2}$. Thus we have

$$
\begin{aligned}
X &= \sum_{i=1}^{n^2} X_i \\
\mathbf{E}[X] &= \sum_{i=1}^{n^2} \mathbf{E}[X_i] \qquad \text{(Linearity of Expectation)} \\
&= \sum_{i=1}^{n^2} \Pr[X_i = 1] \\
&= \sum_{i=1}^{n^2} \frac{1}{n^2} \\
&= 1
\end{aligned}
$$

7. Let T be a tree such that T has at least two vertices and no vertex in T has a degree which is larger than 3. Let n_i be the number of vertices in T of degree exactly i. Prove that

$$n_1 = n_3 + 2$$

Solution. Let n be the total number of vertices in T. Thus, T has $n-1$ edges. We know that

$$n_1 + n_2 + n_3 = n \qquad \text{and} \qquad 2(n-1) = n_1 + 2n_2 + 3n_3 \quad \text{(by Handshaking Lemma)}$$

Combining the two equations, we get

$$\begin{aligned} 2(n_1 + n_2 + n_3 - 1) &= n_1 + 2n_2 + 3n_3 \\ 2n_1 + 2n_2 + 2n_3 - 2 &= n_1 + 2n_2 + 3n_3 \\ n_1 &= n_3 + 2 \end{aligned}$$

Alternate Solution 1. We proceed with a proof by induction on the number of vertices, n. Let $P(n)$ be defined as:

> In any tree with $n \geq 2$ vertices in which no vertex has a degree which is larger than 3, $n_1 = n_3 + 2$.

Base Case: $n = 2$. There is only one such tree. It is a graph with two vertices and an edge between them. Thus, $n_1 = 2$ and $n_3 = 0$ and the claim holds.

Induction Hypothesis: Assume that $P(k)$ is true for some integer $k \geq 2$.

Induction Step: Consider a tree $T = (V, E)$ with $k+1$ vertices. Let ℓ be a leaf in the tree with a neighbor v. We consider the tree $T' = (V \setminus \{\ell\}, E)$ resulting from removing ℓ. Let k_i and k'_i be the number of vertices of degree exactly i in T and T' respectively. By induction hypothesis, $k'_1 = k'_3 + 2$. Now, we add back in ℓ. There are 3 possible cases:

Case 1: v has degree 3 in T and thus degree 2 in T'. Then, $k_1 = k'_1 + 1$ and $k_3 = k'_3 + 1$. From here, $k'_1 + 1 = k'_3 + 2 + 1$ and so $k_1 = k_3 + 2$.

Case 2: v has degree 2 in T and thus degree 1 in T'. Note that in this case, adding back ℓ causes v to no longer be a vertex of degree 1, but also adds back a new vertex of degree 1, ℓ. Thus, $k_2 = k'_2 + 1$, while $k_1 = k'_1$ and $k_3 = k'_3$. From here, $k_1 = k'_1 = k'_3 + 2 = k_3 + 2$.

Case 3: v has degree 1 in T and thus degree 0 in T'. In this case it must be the $k+1 = 2$ and $k = 1$ which is not possible since we bounded k as being at least 2.

In all cases, $k_1 = k_3 + 2$ and so the claim holds for $n = k+1$, thus concluding the proof.

Alternate Solution 2. As proven in homework 8h, the number of leaves in a tree is $2 + \sum_{\substack{v_i \in V, \\ \deg(v_i) \geq 3}} (\deg(v_i) - 2)$. Therefore, we have:

$$n_1 = 2 + \sum_{\substack{v_i \in V, \\ \deg(v_i) \geq 3}} (\deg(v_i) - 2))$$

Observing that the only nodes in T that have degree at least greater than 3 are the nodes with degree 3, we have:

$$\begin{aligned} n_1 &= 2 + \sum_{\substack{v_i \in V, \\ \deg(v_i)=3}} (\deg(v_i) - 2) \\ &= 2 + \sum_{\substack{v_i \in V, \\ \deg(v_i)=3}} (3 - 2) \\ &= 2 + \sum_{\substack{v_i \in V, \\ \deg(v_i)=3}} 1 \\ &= 2 + n_3 \end{aligned}$$

Alternate Solution 3. Let T' be the tree in which each node of degree 2 in T is adjacent to a new node of degree 1. We notice that each node of degree 2 in T becomes a node of degree 3 in T'. We also note that for each node in degree 2, we add one leaf to the graph. Let n'_i be the number of vertices in T' with degree exactly i. It follows that $n'_3 = n_3 + n_2$, $n'_2 = 0$, and $n'_1 = n_1 + n_2$. Therefore, T' is a three-tree. As proven in homework 9h, if a three-tree has ℓ leaves, then it has $\ell - 2$ vertices of degree 3. Therefore, it follows:

$$\begin{aligned} n'_3 &= n'_1 - 2 \\ n_3 + n_2 &= n_1 + n_2 - 2 \\ n_3 &= n_1 - 2 \\ n_1 &= n_3 + 2 \end{aligned}$$

8. Show that if G is connected and the degree of each vertex in G is even then for every vertex $v \in V$, $G - v$ has at most $\frac{\deg(v)}{2}$ connected components. Note that $G - v$ is the graph obtained after removing v and all its incident edges from G.

Solution. After deleting v from G let $CC_1, CC_2, \ldots, CC_k$ be the k connected components in the resulting graph $G - v$. We want to show that $k \leq \deg(v)/2$. Let $N(v)$ denote the set of neighbors of v in G. Since $\deg(v)$ is even so is $|N(v)|$. Observe that in $G - v$ the only odd-degree vertices are the vertices in $N(v)$. Since the number of vertices of odd degree in any graph is even, we know that each connected component CC_i must have at least two vertices from $N(v)$. Thus we have

$$k \leq \frac{|N(v)|}{2} = \frac{\deg(v)}{2}$$

Alternate Solution. Since G is connected and the degree of each vertex in G is even, then we know from lecture that G must have an Eulerian circuit. Let this Eulerian circuit be represented by a sequence of vertices: $v_1, v_2, \ldots v_m, v_1$. WLOG, let v be v_1. We use one edge incident of v when we leave v for the first time and one edge when we enter v for the last time in the Eulerian circuit. And we use two edges for each other time we visit v, since we must enter and leave v. Therefore v will appear exactly $2 + \dfrac{\deg(v) - 2}{2} = \dfrac{\deg(v)}{2} + 1$ times in the Eulerian circuit.

Consider each sequence of vertices in between two consecutive appearances of v in the Eulerian circuit. They are connected by a walk W_i in G, because they are consecutive in the Eulerian circuit. We can rewrite the Eulerian circuit as

$$v \rightarrow W_1 \rightarrow v \rightarrow W_2 \rightarrow \ldots \rightarrow v \rightarrow W_{\deg(v)/2} \rightarrow v$$

All W_i still exist in $G - v$, since v does not appear in any of the W_i. As such, there is a walk in $G - v$ between any two vertices in the same W_i. Although the vertices of $G - v$ may appear in multiple W_i, they all appear in at least one. As such, since there are at most $\deg(v)/2$ distinct W_i in which vertices of $G - v$ could appear, and thus, there are at most $\deg(v)/2$ distinct CC in $G - v$.

Note: We can also make this argument by first arguing that the neighbors of v must be paired into CC and then that every other vertex in $G - v$ must appear on at least one walk between two neighbors of v, giving the same bound on the CC.