CIS 160
Exam 2
November 15, 2018

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a. This exam contains 8 problems. You have 80 minutes to complete the exam.

b. **Last page is scratch paper**. Work on this page will not be graded. Feel free to detach it.

c. The exam is closed-book and closed notes. You are not allowed to use a calculator.

d. Do not spend too much time on any one problem. It may be helpful to first glance through all of them and attack them in the order that allows you to make the most progress.

e. **Unless specified otherwise, you must justify all of your answers. Answers without justification may receive no points.**

f. Unless specified otherwise, you may use any result presented in the class/recitation, or from a homework as a building block for your solutions.

Name: ____________________ **PennKey:** ____________

Recitation Number/TA: ____________________

I certify that I have neither given nor received unauthorized assistance on this exam.

Signature ____________________

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total
/12	/12	/12	/12	/13	/13	/13	/13	/100

[12] 1. AJ decides to choose one of three biased coins (coin A, coin B, coin C) and flip the chosen coin once. Coin A shows heads with probability $\frac{5}{15}$, coin B shows heads with probability $\frac{3}{15}$, and coin C shows heads with probability $\frac{1}{15}$. Suppose that AJ chooses coin A with probability $\frac{1}{4}$, coin B with probability $\frac{1}{4}$ and coin C with probability $\frac{1}{2}$. What is the probability that AJ chose coin C, given that the coin flip resulted in heads? You do not need to simplify your answer.

[12] 2. Answer the following questions. No justification is required. No partial credit will be given for incorrect answers.

a. **Find the flaw or state there is none.**
Claim: For every non-negative integer n, $5n = 0$.

Base Case: $n = 0$. We see $5(0) = 0$ and our claim holds in this case.

Induction Hypothesis: Assume that for some integer k, $5j = 0$ for every integer j, $0 \leq j \leq k$.

Induction Step: Now, we want to prove that $5(k+1) = 0$. First, we write $k+1 = i+j$ where i and j are non-negative integers less than $k+1$. We then note:

$$\begin{aligned} 5(k+1) &= 5(i+j) \\ &= 5i + 5j \\ &= 0 + 0 && \text{(By the IH)} \\ &= 0 \end{aligned}$$

Thus our claim is proven by induction.

b. I shuffle a standard deck of 52 cards thoroughly so that every possible ordering is equally likely. Let E denote the event that the top card is a spade. Let F denote the event that the 3rd card from the top is a spade. We can show that $\Pr[E] = 1/4$. Is $\Pr[F]$ larger than, equal to, or smaller than $1/4$?

c. Let $\Omega = \{a, b, c, d, e\}$ be a uniform sample space. Let A be the event given by $\{a, b, c\}$. Give an example of an event B such that $\Pr[B] > 0$ and A and B are independent.

d. Let $\Omega = \{a, b, c, d\}$ be a uniform sample space. Produce three events A, B, and C that are pairwise independent, but not mutually independent.

[12] 3. An edge $\{u, v\}$ is a *chord* of the cycle C in an undirected simple graph $G = (V, E)$ if u and v are vertices in the cycle, but the edge $\{u, v\} \in E$ is not an edge of the cycle.

Prove the following statement: If $G = (V, E)$ is a simple undirected graph with minimum degree at least 3 then G contains a cycle with a chord.

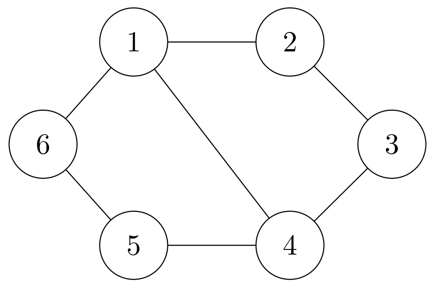

Figure 1: An example of a cycle with chord $\{1, 4\}$.

[12] 4. Suppose that we flip a fair coin until either it comes up tails twice (not necessarily consecutively) or we have flipped it six times. What is the expected number of times we flip the coin? You do not need to simplify your answer.

[13] 5. For any integer $n \geq 3$, let G be a simple undirected graph on n vertices such that for any two vertices u and v in G, it is true that $\deg(u) + \deg(v) \geq n$. Prove that G has a Hamiltonian cycle. For this question only, you may not refer to any lemmas from lecture, recitation or homeworks.

[13] 6. There are n companies and n applicants. Each company has a preference list that ranks every applicant and each applicant has a preference list that ranks each company. There are no ties in the preference lists. We say that a company c and an applicant a form a matching pair if the applicant a is the highest ranked applicant (most preferred applicant) on c's list and if the company c is the highest ranked company (most preferred company) on a's list. Assuming that the preference lists of every company and every applicant are independently and uniformly generated over all permutations of n applicants and over all permutations of n companies, respectively, what is the expected number of matching pairs?

Example. Suppose we have three applicants a_1, a_2, a_3, having the preference lists (from highest ranked to lowest):

a_1: (c_1, c_2, c_3)
a_2: (c_2, c_1, c_3)
a_3: (c_3, c_1, c_2)

and three companies c_1, c_2, c_3, having the preference lists:

c_1: (a_1, a_3, a_2)
c_2: (a_2, a_1, a_3)
c_3: (a_1, a_3, a_2)

Then, we would get two matching pairs, (a_1, c_1) and (a_2, c_2).

[13] 7. Let T be a tree such that T has at least two vertices and no vertex in T has a degree which is larger than 3. Let n_i be the number of vertices in T of degree exactly i. Prove that

$$n_1 = n_3 + 2$$

[13] 8. Show that if G is connected and the degree of each vertex in G is even then for every vertex $v \in V$, $G - v$ has at most $\frac{\deg(v)}{2}$ connected components. Note that $G - v$ is the graph obtained after removing v and all its incident edges from G.

Scratch Paper