CIS 160
Exam 1 Solutions
October 11, 2018

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[34] 1.

(a) Eleven TAs, including Katie, Stephanie, and Nishita, line up for movie tickets. If both Katie and Stephanie are in front of (not necessarily immediately in front of) Nishita, how many possible ways are there for the eleven TAs to stand in line? No justification is necessary, but showing your work may help to get partial credit.

Solution. The procedure for constructing an ordering of the TAs in a line is:

Step 1: Choose 3 spots for Katie, Stephanie, and Nishita, without considering their order.

Step 2: Order Katie, Stephanie, and Nishita within their 3 spots.

Step 3: Arrange the remaining TAs.

Step 1 can be done in $\binom{11}{3}$ ways . There are 2 ways to order Katie, Stephanie, and Nishita. There are 8! ways for the others to stand in line in any order. So by the multiplication rule, the total number of ways is given by

$$\binom{11}{3} \cdot 2 \cdot 8!$$

(b) Give a combinatorial proof for the following identity.

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 3^n$$

Solution. We will prove the identity by answering the following counting question in two different ways.

How many n letter strings can be formed using letters from $\{a, b, c\}$?

Clearly, there are three choices for each of the n letters and hence there are 3^n such strings, which is the RHS.

Another way to count the number of n letter strings that can be formed using letters a, b, and c is as follows. Let S be the set of all such strings. S can be partitioned into $S_0, S_1, S_2, \ldots, S_n$, where S_k is the set of all n letter strings formed using letters a, b, and c that contain exactly k a's. Note that a string in S_k can be formed as follows.

Step 1. Choose the k spots for a.
Step 2. Pick any of b or c for the remaining $n-k$ spots.

Step 1 can be done in $\binom{n}{k}$ ways and step 2 can be done in 2^{n-k} ways. By the multiplication rule, we have $|S_k| = \binom{n}{k}2^{n-k}$. $S_0, S_1, \ldots, S_n$ partition the set S, so:

$$|S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$$

This gives us the LHS.

(c) Prove the identity from part (b) using the binomial theorem.

Solution. The binomial theorem states that for any non-negative integer n and real numbers a ane b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Plugging $a = 2$ and $b = 1$ in the above equation gives us the identity in question.

(d) Find the number of ways in which five different books can be distributed among students Anne, Mary and Dan, if each student gets at least one book and all books must be distributed. No justification is necessary, but showing your work may help in getting partial credit.

Solution. The total number of ways to distribute all books without any constraints $= 3^5$.

The number of ways in which all books go to only one student is $\binom{3}{1} \cdot 1^5 = 3$.

The number of ways in which all books go to two students with exactly one student not getting any book is $\binom{3}{2} \cdot (2^5 - 2) = 90$.

Therefore, our answer is $3^5 - 3 - 90 = 150$.

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[10] 2. Let A, B, and C be any arbitrary sets. Show that

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

Solution. Let (x, y) be an arbitrary but particular element of the LHS, i.e., $(x, y) \in (A \times B) \cap (A \times C)$. This means that (x, y) belongs to both $A \times B$ and $A \times C$. Since $(x, y) \in A \times B$, we have $x \in A$. Since $(x, y) \in A \times B$ and $(x, y) \in A \times C$, we have $y \in B$ and $y \in C$, that is, $y \in B \cap C$. Thus $(x, y) \in A \times (B \cap C)$.

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[10] 3. For all integers $x \geq 0$, let $G_x = 2^{2^x} + 1$. Prove by induction that for all integers $n \geq 1$,

$$\prod_{i=0}^{n-1} G_i = G_n - 2 \tag{1}$$

(The symbol $\prod$ in the above equation stands for product. Thus $\prod_{i=0}^{n-1} G_i = G_0 \times G_1 \times \cdots \times G_{n-1}$). Show the work to justify your answer.

Solution. We will prove the claim using induction on n.

Induction Hypothesis: Assume that the claim is true for some integer $k \geq 1$. In other words, assume that

$$\prod_{i=0}^{k-1} G_i = G_k - 2$$

Base Case: When $n = 1$, the left hand side of 1 equals $G_0 = 2^{2^0} + 1 = 3$. The right hand side equals $G_1 - 2 = 2^{2^1} + 1 - 2 = 5 - 2 = 3$. Thus the claim holds when $n = 1$.

Induction Step: We want to prove that the claim holds when $n = k + 1$. That is, we want to show that

$$\prod_{i=0}^{k} G_i = G_{k+1} - 2$$

The left hand side of the above equation is given by

$$\begin{aligned}
\prod_{i=0}^{k} G_i &= \left(\prod_{i=0}^{k-1} G_i\right) G_k \\
&= (G_k - 2)G_k && \text{(using Induction Hypothesis)} \\
&= (2^{2^k} + 1 - 2)(2^{2^k} + 1) \\
&= (2^{2^k} - 1)(2^{2^k} + 1) \\
&= (2^{2^k})^2 - 1 \\
&= 2^{2 \cdot 2^k} + 1 - 2 \\
&= 2^{2^{k+1}} + 1 - 2 \\
&= G_{k+1} - 2
\end{aligned}$$

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[10] 4. Prove that if $n \in \mathbb{Z}^+$ and $a_1, a_2, \ldots, a_n, a_{n+1}$ are positive integers (not necessarily distinct) then there is a pair (a_i, a_j) such that $i \neq j$ and $a_i - a_j$ is divisible by n.

Solution. There are n possible remainders when a positive integer is divided by n, namely, $0, 1, \ldots, n-1$. Let the pigeons be the $n+1$ numbers, and the holes be the n possible remainders. By the pigeonhole principle, two of these numbers must have the same remainder when divided by n. Let the two numbers be a_i and a_j, where $i \neq j$. Thus, for some $r \in \{0, 1, 2, \ldots, n-1\}$, we have

$$a_i = nq + r$$
$$a_j = nq' + r$$

Thus we have $a_i - a_j = n(q - q')$, so $n \mid a_i - a_j$.

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[12] 5. Prove that between any two distinct rational numbers, there exist infinitely many rational numbers.

Solution. Assume for contradiction that between two rational numbers, say x and y, where (WLOG) $x < y$, there exist only finitely many rational numbers; let S denote this set. In other words, S is the set of rational numbers in the interval (x, y). Let z be the largest rational number in S. Consider the number $r = (z + y)/2$. Clearly, r is rational. Also, $z < r < y$. Then r must be in S and must have been the largest rational number in S, a contradiction.

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[12] 6. Let p and q be positive integers. Consider the set of all binary strings containing exactly p 0's and exactly q 1's. Show that exactly $\binom{p-q+2}{q}$ of these strings have at least two 0's between every pair of 1's. You may assume that $p \geq 2(q-1)$. Justify your answer.

Solution. The procedure for constructing an arrangement of p 0's and q 1's with the given constraints is as follows.

> *Step 1.* Arrange all the q 1's.
> *Step 2.* Let $s_0, s_1, \ldots, s_q$ be the $q+1$ spots, where $s_i, 1 \leq i < q$, is the spot between the ith and the $(i+1)$th 1 and s_0 and s_q are spots before the start of the first 1 and after the last 1 respectively. Let x_i be the number of 0's in spot i. Find a solution to the equations $x_i \geq 2, 1 \leq i \leq q-1$, and $\sum_{i=0}^{q} x_i = p$.

There is one way to do Step 1. In Step 2, after giving away the two 0's to each of the $q-1$ positions, the remaining $p - 2(q-1) = (p - 2q + 2)$ 0's can be distributed among the $q+1$ slots using the sticks and crosses method in which the number of sticks equals q and the number of crosses equal $p - 2q + 2$. Thus, Step 2 can be done in

$$\binom{q + p - 2q + 2}{q} = \binom{p - q + 2}{q}$$

ways. By the multiplication rule, the total number of strings is

$$1 \times \binom{p-q+2}{q} = \binom{p-q+2}{q}$$

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[12] 7. Let S be the set of all binary strings. Let $y \cdot z$ denote the concatenation of the strings y and z. Thus if $y = 00$ and $z = 10$ then $y \cdot z = 0010$. Similarly, if y is an empty string and $z = 11$ then $y \cdot z = 11$.

Prove that for all integers $n \geq 0$, any string $x \in S$ of length n can be written in the form $x = y \cdot z$ where the number of 0's in y is the same as the number of 1's in z. Empty strings (strings of length 0) are allowed. For example, the string 010010 can be written as $01 \cdot 0010$ and the string 11101000 can be written as $1110 \cdot 1000$.

Solution. Proof by induction on n. We first define $n_0(s)$ to denote the number of 0's in a string s, and $n_1(s)$ to denote the number of 1's in a string s. Let $P(n)$ be the property that any binary string x of length n can be written as a concatenation of strings y and z such that $n_0(y) = n_1(z)$; that is, the number of 0's in y equals the number of 1's in z.

Induction Hypothesis: Assume that $P(k)$ holds for some integer $k \geq 0$.

Base Case ($n = 0$): x is an empty string and can be written as $x = y \cdot z$, where y and z are both empty strings as well. $n_0(y) = 0 = n_1(z)$, so $P(0)$ holds.

Induction Step: We want to prove that $P(k+1)$ holds. Let $x = x_1 x_2 \ldots x_k x_{k+1}$ be a binary string of length $k+1$. Consider the substring $x' = x_1 x_2 \ldots x_k$. By the Induction Hypothesis, x' can be written as $y' \cdot z'$, where $n_0(y') = n_1(z')$. Then,

$$x = x' \cdot x_{k+1} = y' \cdot z' \cdot x_{k+1}$$

We now case on the value of x_{k+1}, the last character in x.

Case 1 ($x_{k+1} = 0$): We can write x as $y \cdot z$, where $y = y'$ and $z = z' \cdot x_{k+1}$ and the claim holds since

$$n_0(y) = n_0(y') = n_1(z') = n_1(z)$$

Case 2 ($x_{k+1} = 1$): Let $z' = z'_1 z'_2 \ldots z'_p$. We can write x as $y \cdot z$, where $y = y' \cdot z'_1$ and $z = z'_2 \ldots z'_p \cdot x_{k+1}$. In other words, we shift our original partition of the string x to the right by 1 character. We can show that $n_0(y)$ equals $n_1(z)$ by considering the following subcases on the value of z'_1:

Case a ($z'_1 = 0$): We want to show that $n_0(y' \cdot z'_1) = n_1(z'_2 \dots z'_p \cdot x_{k+1})$.

$$\begin{aligned} n_0(y' \cdot z'_1) &= n_0(y' \cdot 0) \\ &= n_0(y') + 1 \\ &= n_1(z') + 1 && \text{(by the Induction Hypothesis)} \\ &= n_1(z'_2 \dots z'_p \cdot x_{k+1}) \end{aligned}$$

The last steps hold since $z'_1 = 0$ and $x_{k+1} = 1$, so we gain a 0 in y from z'_1 and gain a 1 in z from x_{k+1}.

Case b ($z'_1 = 1$): We want to show that $n_0(y' \cdot z'_1) = n_1(z'_2 \dots z'_p \cdot x_{k+1})$.

$$\begin{aligned} n_0(y' \cdot z'_1) &= n_0(y' \cdot 1) \\ &= n_0(y') \\ &= n_1(z') && \text{(by the Induction Hypothesis)} \\ &= n_1(z'_2 \dots z'_p \cdot x_{k+1}) \end{aligned}$$

The last steps hold since $z'_1 = 1$ and $x_{k+1} = 1$, so we are essentially losing a 1 and gaining a 1 from z' to $z = z'_2 \dots z'p \cdot x_{k+1}$.

We have shown that the claim holds in all possible cases, and this completes the proof of the induction step.

Alternate: Direct.

Let $n_0(s)$ be the number of 0's in a binary string s and let $n_1(s)$ be the number of 1's in s.

Consider the partition $x = y{\cdot}z$ where y is the empty string and $z = x$. Clearly, $n_0(y) = 0$ and $n_1(z) \geq 0$. This leaves us with 2 possible cases: either $n_1(z) - n_0(y) = 0$ or $n_1(z) - n_0(y) \geq 1$. In the former case, we are done as we have found a partition such that $n_0(y) = n_1(z)$. We now consider the latter case.

Consider an arbitrary partition such that $x = y \cdot z$ where $y = y_1 \dots y_p$ and $z = z_1 \dots z_k$ (y and z are non-empty). When we move the partition one bit to the right, we get a new partition: $x = y' \cdot z'$ where $y' = y_1 \dots y_p z_1$ and $z' = z_2 \dots z_k$. There are 2 possible cases to consider, either $z_1 = 0$ or $z_1 = 1$.

Case 1: $z_1 = 0$. In this case, $n_0(y') = n_0(yz_1) = n_0(y \cdot 0) = n_0(y) + 1$ and $n_1(z') = n_1(z_2 \dots z_k) = n_1(z)$ (since $z_1 = 0$). Plugging in gives us, $n_1(z') - n_0(y') = n_1(z) - (n_0(y) + 1) = n_1(z) - n_0(y) - 1$.

Case 2: $z_1 = 1$. In this case, $n_0(y') = n_0(y_1 \dots y_p z_1) = n_0(y_1 \dots y_p \cdot 1) = n_0(y)$ and $n_1(z') = n_1(z_2 \dots z_k) = n_1(z) - 1$ (since $z_1 = 1$). Plugging in gives us, $n_1(z') - n_0(y') = n_1(z) - 1 - n_0(y) = n_1(z) - n_0(y) - 1$.

In both cases, we get $n_1(z') - n_0(y') = n_1(z) - n_0(y) - 1$. From here, we can see that moving the partition one bit to the right will always decrease the difference between the 0's in y and the 1's in z by exactly 1.

Finally, consider the partition all the way on the right, i.e. $x = y \cdot z$ where $x = y$ and z is the empty string. Here, $n_1(z) - n_0(y) = 0 - n_0(y) = -n_0(y)$. Clearly, $n_0(y) \geq 0$ and so $-n_0(y) \leq 0$.

Thus, we begin with a partition all the way on the left such that $n_1(z) - n_0(y) \geq 0$. We know that shifting the partition one bit to the right decreases $n_1(z) - n_0(y)$ by 1 and that eventually we reach a partition all the way on the right such that $n_1(z) - n_0(y) \leq 0$. From here, it follows that there must exists a partition in which $n_1(z) - n_0(y) = 0$ and the claim is proven.