

CIS 160
Exam 1
October 11, 2018

- a. This exam contains 7 problems. You have 80 minutes to complete the exam.
- b. **Last page is scratch paper.** Work on this page will not be graded. Feel free to detach it.
- c. The exam is closed-book and closed notes. You are not allowed to use a calculator.
- d. Do not spend too much time on any one problem. It may be helpful to first glance through all of them and attack them in the order that allows you to make the most progress.
- e. **Unless specified otherwise, you must justify all of your answers. Answers without justification may receive no points.**
- f. You may use any result presented in the class/recitation, or from a homework as a building block for your solutions.

Name: _____ PennKey: _____

Recitation Number/TA: _____

I certify that I have neither given nor received unauthorized assistance on this exam.

Signature _____

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
/34	/10	/10	/10	/12	/12	/12	/100

[34] 1.

- (a) Eleven TAs, including Katie, Stephanie, and Nishita, line up for movie tickets. If both Katie and Stephanie are in front of (not necessarily immediately in front of) Nishita, how many possible ways are there for the eleven TAs to stand in line? No justification is necessary, but showing your work may help to get partial credit.

(b) Give a combinatorial proof for the following identity.

$$\sum_{k=0}^n \binom{n}{k} 2^{n-k} = 3^n$$

- (c) Prove the identity from part (b) using the binomial theorem.

- (d) Find the number of ways in which five different books can be distributed among students Anne, Mary and Dan, if each student gets at least one book and all books must be distributed. No justification is necessary, but showing your work may help in getting partial credit.

[10] **2.** Let A, B , and C be arbitrary sets. Show that

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

[10] 3. For all integers $x \geq 0$, let $G_x = 2^{2^x} + 1$. Prove by induction that for all integers $n \geq 1$,

$$\prod_{i=0}^{n-1} G_i = G_n - 2$$

(The symbol \prod in the above equation stands for product. Thus $\prod_{i=0}^{n-1} G_i = G_0 \times G_1 \times \cdots \times G_{n-1}$). Show the work to justify your answer.

[10] 4. Prove that if $n \in \mathbb{Z}^+$ and $a_1, a_2, \dots, a_n, a_{n+1}$ are positive integers (not necessarily distinct) then there is a pair (a_i, a_j) such that $i \neq j$ and $a_i - a_j$ is divisible by n .

[12] 5. Prove that between any two distinct rational numbers, there exist infinitely many rational numbers.

[12] **6.** Let p and q be positive integers. Consider the set of all binary strings containing exactly p 0's and exactly q 1's. Show that exactly $\binom{p-q+2}{q}$ of these strings have at least two 0's between every pair of 1's. You may assume that $p \geq 2(q-1)$. Justify your answer.

[12] 7. Let S be the set of all binary strings. Let $y \cdot z$ denote the concatenation of the strings y and z . Thus if $y = 00$ and $z = 10$ then $y \cdot z = 0010$. Similarly, if y is an empty string and $z = 11$ then $y \cdot z = 11$.

Prove that for all integers $n \geq 0$, any string $x \in S$ of length n can be written in the form $x = y \cdot z$ where the number of 0's in y is the same as the number of 1's in z . Empty strings (strings of length 0) are allowed. For example, the string 010010 can be written as $01 \cdot 0010$ and the string 11101000 can be written as $1110 \cdot 1000$.

Scratch Paper