Mathematical Foundations of Computer Science
Solutions to Practice Problems for Exam 2

P1: Let $a_0 = 1$. Suppose $a_{n+1} = 2 \cdot \sum_{i=0}^{n} a_i$. Find an explicit formula for a_n and prove your claim by strong induction. (Here, explicit means that you can compute a_n knowing just the value of n and nothing else.)

Solution. We begin by evaluating several values of a_n:

n	a_n
0	1
1	2
2	6
3	18
4	54
5	162
6	486

We notice that each successive value of an increases by a multiple of 3. Thus, after some testing with the coefficient, we reach $a_n = 2 \times 3^{n-1}$. This formula applies to all values of a_n EXCEPT for a_0. Thus, we instead use a piecewise formula:

$$\forall n \in \mathbb{N}, a_n = \begin{cases} 1 & n = 0 \\ 2 \times 3^{n-1} & n > 0 \end{cases}$$

We now perform strong induction to prove that our formula holds for all n:
Let $P(n)$ be defined as

$$a_n = 2 \cdot \sum_{i=0}^{n-1} a_i = \begin{cases} 1 & n = 0 \\ 2 \times 3^{n-1} & n > 0 \end{cases}$$

Induction hypothesis: Assume that $P(j)$ is true for all $0 \leq j \leq k$, for some $k \geq 0$.
Base Case: $P(0)$ is true because $P(0) \equiv a_0 = 1$ by definition.
$P(1)$ is true because:

$$\begin{aligned} LHS = a_1 &= 2 \cdot \sum_{i=0}^{0} a_i \\ &= 2 \cdot 1 \\ &= 2 \\ RHS &= 2 \times 3^{1-1} \\ &= 2 \times 1 \\ &= 2 \end{aligned}$$

Induction Step: We want to show that $P(k+1)$ is true, such that:

$$a_{k+1} = 2 \cdot \sum_{i=0}^{(k+1)-1} a_i = 2 \times 3^{(k+1)-1} = 2 \times 3^k$$

$$\begin{aligned} LHS = a_{k+1} &= 2 \cdot \sum_{i=0}^{(k+1)-1} a_i \\ &= 2\left[a_0 + \sum_{i=1}^{k} a_i\right] \end{aligned}$$

Invoking the induction hypothesis $a_j = 2 \times 3^{j-1}$:

$$= 2\left[1 + \sum_{i=1}^{k} 2 \times 3^{i-1}\right]$$

Using the sum of geometric series to further reduce:

$$\begin{aligned} &= 2\left[1 + 2 \times \frac{1-3^k}{1-3}\right] \\ &= 2\left[1 + 2 \times \frac{3^k - 1}{2}\right] \\ &= 2[1 + 3^k - 1] \\ &= 2 \times 3^k = RHS \end{aligned}$$

This completes the proof.

P2: I dip a $3 \times 3 \times 3$ cube into paint so its entire surface is coated. I then disassemble the cube into 27 cubelets (of size $1 \times 1 \times 1$), take one randomly, and place it in front of you on a table. From the five sides you can observe of the cubelet, no side is painted. What is the probability that the bottom side (that you cannot observe) is painted?

Solution. Let the cube be C. Clearly the cubelet in front of you has no sides painted or exactly one side (the bottom side) painted. There are only seven cubelets that have this property (six cubelets on the center of each face of C, one cubelet at the center of C). However, each of these cubelets is not equally likely to be in front of you at this point. This is because if I picked up the center cubelet (no painted side), I could have placed it down on any of its six faces, and you would see all observable sides not painted. But if I picked up any of the other six cubelets, I have to place them down in exactly one way: with the painted side on the bottom. Thus, there are twelve total ways (outcomes), all equally likely, for me to pick up and place a cubelet in front of you when we distinguish different orientations of the cube. Since six of these twelve outcomes involved a cubelet with paint on the bottom side, the probability that the bottom has paint, and thus the answer to this question is $\frac{1}{2}$.

We can also do this problem with familiar notation. We also consider the following events.

A: event that the bottom side of the cubelet in front of you is painted.
B: event that all five observable sides of the cubelet are not painted.

Note that $A \cap B$ represents the event that you pick up one of the six cubelets with one side painted and also place it with its painted side down. Then:

$$\begin{aligned}\Pr[A|B] = \frac{\Pr[A\cap B]}{\Pr[B]} &= \frac{\Pr[A\cap B]}{\Pr[B\cap A] + \Pr[B\cap \overline{A}]} \\ &= \frac{\frac{6}{27}\times\frac{1}{6}}{\frac{6}{27}\times\frac{1}{6}+\frac{1}{27}\times\frac{6}{6}} \\ &= \frac{1}{2}\end{aligned}$$

P3: Let G be a connected graph where all vertices are of even degree. Prove that G has no *cut edges*. A *cut edge* is an edge, that if removed, would increase the number of connected components of the graph.

Solution. Suppose, for the sake of contradiction, that G does have a cut edge $e = \{v_1, v_2\}$. Since G is connected, $G - e$ has exactly two connected components. Note that each vertex in $G - e$ has the same degree as in G except v_1 and v_2, whose degrees are each one less than in G. Thus, v_1 is the only vertex of odd degree in its connected component in $G - e$. Therefore, there are an odd number of odd-degree vertices in that connected component. However, we know that there must be an even number of odd degreed vertices because each connected component of a graph is itself a graph, and hence contains even number of odd degree vertices. Thus, G has no cut edges.

P4: Let $T = (V, E)$ be a tree with $n \geq 2$ vertices. Prove that for any vertex $u \in V$,

$$\sum_{v\in V} d(u,v) \leq \binom{n}{2}$$

Solution. The proof is by induction on n.
Induction hypothesis: Suppose that for all trees with k vertices (for some $k \geq 2$), for any vertex u, the inequality holds.
Base Case: When $n = 2$, we can write $V = \{u_0, u_1\}$. Without loss of generality, let $u = u_0$. Then:

$$d(u_0, u_0) + d(u_0, u_1) = 0 + 1 = 1 = \binom{2}{2}$$

Induction Step: Let $T = (V, E)$ be a tree with $k + 1$ vertices. Fix a leaf $x \in V$ and let $T' = T - x$ be the subtree. Now let u be an arbitrary vertex of V. We have two cases:

- Suppose $u \neq x$, which means that u is a vertex in T'. By the inductive hypothesis applied to T', it is the case that

$$\sum_{v \in V - \{x\}} d(u, v) \leq \binom{k}{2}$$

Notice that

$$\sum_{v \in V} d(u, v) = d(u, x) + \sum_{v \in V - \{x\}} d(u, v) \leq d(u, x) + \binom{k}{2}$$

Since there are k vertices in $T - x$, it must be the case that $d(u, x) \leq k$. Thus

$$\sum_{v \in V} d(u, v) \leq k + \binom{k}{2}$$

Recall that $\binom{k}{2} = \frac{k(k-1)}{2}$, which means that $k + \binom{k}{2} = \frac{k(2+k-1)}{2} = \frac{k(k+1)}{2} = \binom{k+1}{2}$. Thus

$$\sum_{v \in V} d(u, v) \leq k + \binom{k}{2} = \binom{k+1}{2}$$

- Suppose $u = x$. Since x is a leaf, there is a unique vertex y such that (x, y) is an edge in T. Notice then that

$$\sum_{v \in V} d(x, v) = d(x, x) + \sum_{v \in V - \{x\}} d(u, v) = 0 + \sum_{v \in V - \{x\}} (1 + d(y, v)) = k + \sum_{v \in V - \{x\}} d(y, v)$$

By the inductive hypothesis we know that $\sum_{v \in V - \{x\}} d(y, v) \leq \binom{k}{2}$, and therefore

$$\sum_{v \in V} d(x, v) \leq k + \binom{k}{2} = \binom{k+1}{2}$$

as before.

P5: A CIS160 angel tells you in a dream that every connected graph has a connected subgraph that is a tree, which retains all the vertices of the original graph (called a *spanning tree*). The angel also tells you a procedure that allows you to find that exact subgraph given any connected graph, G. The following is a procedure: We will keep adding edges to a subgraph H of G so that at the end H is a spanning tree of G. Initially H has no edges and $V(H) := V(G)$. While H has more than 1 component, find an edge in G that has endpoints in two different components of H and add it to H. Prove the following properties:

A. If H has more than 1 component, there is some edge in G whose endpoints lie in different components of H.

B. At all times H is an acyclic graph.

C. When this procedure terminates, H will be a spanning tree of G.

Solution.

A. Suppose H has more than one component and each edge in G has endpoints in the same component of H. Then there are no paths in G between vertices in different components of H, contradicting the fact that G is connected. So we must have edges in G with endpoints in different components.

B. The proof is by induction on the number of edges n added to H.

Induction hypothesis: The subgraph H is acyclic after the addition of $n = k$ edges, for some $k \geq 0$.
Base Case: When $n = 0$, the subgraph H has no edges in it, and therefore is acyclic.
Induction Step: Consider the subgraph H after the addition of $k+1$ edges. Note that by the induction hypothesis, the subgraph was acyclic prior to the addition of the $k+1^{th}$ edge. Now let us consider the addition of the $k+1^{th}$ edge (u, v). Suppose for contradiction that (u, v) causes a cycle to be formed in H. This cycle must include the edge (u, v) because there was no cycle before it was added. Then before the addition of (u, v), there must have been a path in H between u and v. However, u and v were in different components of H, meaning that there was no such path, a contradiction. Thus the addition of the $k+1^{th}$ edge does not cause a cycle and we have proved the inductive step.
Remark: Note that we use a proof by induction here even though we are not proving a statement for an infinite set of cases. We will only add finitely many edges to H, but the proof by induction is still valid.

C. As long as H contains more than one component, we will find an edge in G to add to H by part (a). Thus we will only stop when H becomes a single component, at which point it will be connected and acyclic by part (b), i.e., a tree.