# Mathematical Foundations of Computer Science
# Practice Problems for Exam 2

---

**P1:** Let $a_0 = 1$. Suppose $a_{n+1} = 2 \cdot \sum_{i=0}^{n} a_i$. Find an explicit formula for $a_n$ and prove your claim by strong induction. (Here, explicit means that you can compute $a_n$ knowing just the value of $n$ and nothing else.)

**P2:** I dip a $3 \times 3 \times 3$ cube into paint so its entire surface is coated. I then disassemble the cube into 27 cubelets (of size $1 \times 1 \times 1$), take one randomly, and place it in front of you on a table. From the five sides you can observe of the cubelet, no side is painted. What is the probability that the bottom side (that you cannot observe) is painted?

**P3:** Let $G$ be a connected graph where all vertices are of even degree. Prove that $G$ has no *cut edges*. A *cut edge* is an edge, that if removed, would increase the number of connected components of the graph.

**P4:** Let $T = (V, E)$ be a tree with $n \geq 2$ vertices. Prove that for any vertex $u \in V$,

$$\sum_{v \in V} d(u, v) \leq \binom{n}{2}$$

**P5:** A CIS160 angel tells you in a dream that every connected graph has a connected subraph that is a tree, which retains all the vertices of the original graph (called a *spanning tree*). The angel also tells you a procedure that allows you to find that exact subgraph given any connected graph, $G$. The following is a procedure: We will keep adding edges to a subgraph $H$ of $G$ so that at the end $H$ is a spanning tree of $G$. Initially $H$ has no edges and $V(H) := V(G)$. While $H$ has more than 1 component, find an edge in $G$ that has endpoints in two different components of $H$ and add it to $H$. Prove the following properties:

**A**. If $H$ has more than 1 component, there is some edge in $G$ whose endpoints lie in different components of $H$.

**B**. At all times $H$ is an acyclic graph.

**C**. When this procedure terminates, $H$ will be a spanning tree of $G$.